$\boxed{18.1}$

a) $$u + iv = (a+ib)(c+id)$$
$$= (ac-bd) + i(ad+bc)$$

então: $$u = ac - bd \Rightarrow u^2 = (ac)^2 + (bd)^2 - 2abcd$$
$$v = ad + bc \Rightarrow v^2 = (ad)^2 + (bc)^2 + 2abcd$$

$$u^2 + v^2 = (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2$$
$$= a^2(c^2+d^2) + b^2(c^2+d^2)$$

$$u^2 + v^2 = (c^2+d^2)(a^2+b^2)$$

$$\sqrt{u^2+v^2} = \sqrt{(a^2+b^2)(c^2+d^2)}$$

b) $$\frac{v}{u} = \frac{ad+bc}{ac-bd} = \frac{ad}{ac-bd} + \frac{bc}{ac-bd} = \frac{a\frac{d}{c}}{a - b\frac{d}{c}} + \frac{\frac{b}{a}c}{c - \frac{b}{a}}$$

$$\frac{v}{u} = \frac{\frac{a}{b}\cdot\frac{d}{c}}{\frac{a}{b} - \frac{d}{c}} + \frac{\frac{b}{a}\frac{c}{d}}{\frac{c}{d} - \frac{b}{a}}$$ Mas: $\tan\alpha = \frac{b}{a}$; $\tan\beta = \frac{d}{c}$

Então:
$$\frac{v}{u} = \frac{\frac{\tan\beta}{\tan\alpha}}{\frac{1}{\tan\alpha} - \tan\beta} + \frac{\frac{\tan\alpha}{\tan\beta}}{\frac{1}{\tan\beta} - \tan\alpha}$$

$$= \frac{\frac{\tan\beta}{\cancel{\tan\alpha}}}{\frac{1 - \tan\alpha\tan\beta}{\cancel{\tan\alpha}}} + \frac{\frac{\tan\alpha}{\cancel{\tan\beta}}}{\frac{1 - \tan\alpha\tan\beta}{\cancel{\tan\beta}}}$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad (\text{eq. } 1)$$

$$\tan(\alpha+\beta) = \frac{\operatorname{sen}(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\operatorname{sen}\alpha\cos\beta + \operatorname{sen}\beta\cos\alpha}{\cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta}$$

$$= \frac{\operatorname{sen}\alpha\cos\beta}{\cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta} + \frac{\operatorname{sen}\beta\cos\alpha}{\cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta}$$

$$= \frac{\frac{\operatorname{sen}\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}}}{1-\frac{\operatorname{sen}\alpha\operatorname{sen}\beta}{\cos\alpha\cos\beta}} + \frac{\frac{\operatorname{sen}\beta\cancel{\cos\alpha}}{\cancel{\cos\alpha}\cos\beta}}{1-\frac{\operatorname{sen}\alpha\operatorname{sen}\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \quad \text{(eq. 2)}$$

Comparando as equações (1) e (2):

$$\frac{v}{u} = \tan(\alpha+\beta)$$

$\boxed{18.2}$

$$\left.\begin{array}{l} u = r_1\cos\theta \\ v = r_1\operatorname{sen}\theta \end{array}\right\} \Rightarrow r_1 = \sqrt{u^2+v^2} \; ; \; \left.\begin{array}{l} a = r_2\cos\alpha \\ b = r_2\operatorname{sen}\alpha \end{array}\right\} \Rightarrow r_2 = \sqrt{a^2+b^2} \; ; \; \left.\begin{array}{l} c = r_3\cos\beta \\ d = r_3\operatorname{sen}\beta \end{array}\right\} \Rightarrow r_3 = \sqrt{c^2+d^2}$$

$$r_1(\cos\theta + i\operatorname{sen}\theta) = r_2(\cos\alpha + i\operatorname{sen}\alpha)\, r_2(\cos\beta + i\operatorname{sen}\beta)$$

$$r_1 e^{i\theta} = r_2 e^{i\alpha} r_3 e^{i\beta} \Rightarrow r_1 e^{i\theta} = r_2 r_3 e^{i(\alpha+\beta)}$$

Então: $r_1 = r_2 r_3$ e $\theta = \alpha+\beta$

a) $\sqrt{u^2+v^2} = \sqrt{a^2+b^2}\sqrt{c^2+d^2}$ c.q.d.

b) $\frac{v}{u} = \frac{r_1\operatorname{sen}\theta}{r_1\cos\theta} = \tan\theta = \tan(\alpha+\beta)$ c.q.d

$\boxed{18.3}$

$$e^{i\theta} = \cos\theta + i\,\mathrm{sen}\,\theta$$
$$e^{-i\theta} = \cos\theta - i\,\mathrm{sen}\,\theta$$

a) $e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

b) $e^{i\theta} - e^{-i\theta} = i\,2\,\mathrm{sen}\,\theta \Rightarrow \mathrm{sen}\,\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\boxed{18.4}$ $\dfrac{a+ib}{c+id} = \dfrac{(a+ib)(c-id)}{(c+id)(c-id)} = \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$\boxed{18.5}$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \quad e \quad \mathrm{senh}\,\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

- $\cos(i\theta) = \dfrac{e^{i(i\theta)} + e^{-i(i\theta)}}{2} = \dfrac{e^{\theta} + e^{-\theta}}{2} = \cosh\theta$
- $\mathrm{sen}(i\theta) = \dfrac{e^{i(i\theta)} - e^{-i(i\theta)}}{2i} = \dfrac{-(e^{\theta} - e^{-\theta})}{2i}$

$$= i\frac{(e^{\theta}-e^{-\theta})}{2} = i\sinh(\theta)$$

- $\cosh^2\theta - \sinh^2\theta = \left(\dfrac{e^{\theta}+e^{-\theta}}{2}\right)^2 - \left(\dfrac{e^{\theta}-e^{-\theta}}{2}\right)^2$

$$\cosh^2\theta - \sinh^2\theta = \cancel{\frac{e^{2\theta}}{4}} + \cancel{\frac{e^{-2\theta}}{4}} + \frac{e^{\theta}e^{-\theta}}{2} - \cancel{\frac{e^{2\theta}}{4}} + \cancel{\frac{e^{-2\theta}}{4}} + \frac{e^{\theta}e^{-\theta}}{2}$$
$$= e^{\theta}e^{-\theta} = 1$$

18.6

$$\frac{d}{dx}e^{\alpha x}=\lim_{\Delta x\to 0}\frac{e^{\alpha(x+\Delta x)}-e^{\alpha x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{\alpha x}(e^{\alpha\Delta x}-1)}{\Delta x}$$

Façamos a seguinte mudança de variável: $e^{\alpha \Delta x}-1=a$

$$\alpha\Delta x \ln e=\ln(1+a)\Rightarrow \alpha\Delta x=\ln(1+a)$$

$$\Delta x=\frac{\ln(1+a)}{\alpha}$$

Notar que $\Delta x\to 0\Rightarrow a\to 0$. Então:

$$\frac{d}{dx}e^{\alpha x}=\lim_{a\to 0}e^{\alpha x}\frac{a}{\frac{\ln(1+a)}{\alpha}}=\alpha e^{\alpha x}\lim_{a\to 0}\frac{a}{\ln(1+a)}$$

$$=\alpha e^{\alpha x}\lim_{a\to 0}\frac{1}{\frac{1}{a}\ln(1+a)}$$

Façamos nova mudança de variável:

$b=\frac{1}{a}$. Notar que quando $a\to 0\Rightarrow b\to\infty$

$$\frac{d}{dx}e^{\alpha x}=\alpha e^{\alpha x}\lim_{b\to\infty}\frac{1}{b\ln\left(1+\frac{1}{b}\right)}$$

$$=\alpha e^{\alpha x}\lim_{b\to\infty}\frac{1}{\ln\left(1+\frac{1}{b}\right)^{b}}$$

$$=\alpha e^{\alpha x}\frac{1}{\ln\lim_{b\to\infty}\left(1+\frac{1}{b}\right)^{b}}$$

Mas $e\equiv\lim_{b\to\infty}\left(1+\frac{1}{b}\right)^{b}$. Então:

$$\frac{d}{dx}e^{\alpha x}=\alpha e^{\alpha x}\frac{1}{\ln e}\Rightarrow\frac{d}{dx}e^{\alpha x}=\alpha e^{\alpha x}\quad \text{c.q.d}$$

$\boxed{18.7}$

A série de Taylor para $f(x)$ no ponto $x = x_0$ é:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

Se $x_0 = 0$ temos a série de Mac-Laurin

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)\, x^n}{n!}$$

a)
$$f(x) = e^x \Rightarrow f(0) = 1$$
$$f'(x) = e^x \Rightarrow f'(0) = 1$$
$$f''(x) = e^x \Rightarrow f''(0) = 1$$
$$f'''(x) = e^x \Rightarrow f'''(0) = 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

b)
$$f(x) = \cos x \Rightarrow f(0) = 1$$
$$f'(x) = -\operatorname{sen} x \Rightarrow f'(0) = 0$$
$$f''(x) = -\cos x \Rightarrow f''(0) = -1$$
$$f'''(x) = \operatorname{sen} x \Rightarrow f'''(0) = 0$$
$$f^{(IV)}(x) = \cos x \Rightarrow f^{(IV)}(0) = 1$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots$$

$$f(x) = \operatorname{sen} x \Rightarrow f(0) = 0$$
$$f'(x) = \cos x \Rightarrow f'(0) = 1$$
$$f''(x) = -\operatorname{sen} x \Rightarrow f''(0) = 0$$
$$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$$
$$f^{(IV)}(x) = \operatorname{sen} x \Rightarrow f^{(IV)}(0) = 0$$

$$\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots$$

18.8

O teorema de Moivre estabelece que se p é um um número real, então:

$$[r(\cos\theta + i\,\text{sen}\,\theta)]^p = r^p[\cos(p\theta) + i\,\text{sen}(p\theta)].$$

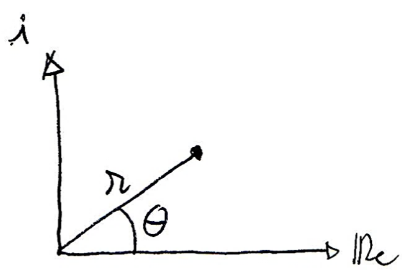
Se, em particular, $p = \frac{1}{n}$, então a equação anterior pode ser escrita como:

$$[r(\cos\theta + i\,\text{sen}\,\theta)]^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\,\text{sen}\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

onde k é qualquer número inteiro.

$$y = \sqrt[n]{1} \Rightarrow y = 1^{1/n} \Rightarrow y = 1^{1/n}\left[\cos\left(\frac{2\pi k}{n}\right) + i\,\text{sen}\left(\frac{2\pi k}{n}\right)\right]$$

$$y = \cos\left(\frac{2\pi k}{n}\right) + i\,\text{sen}\left(\frac{2\pi k}{n}\right)$$ c.q.d.



$\boxed{18.9}$

Teo. binomial:

$$(x+y)^n = x^n + n x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + \cdots + y^n$$

Teo. de Moivre:

$$r\left[\cos\theta + i \operatorname{sen}\theta\right]^n = r^n\left[\cos(n\theta) + i \operatorname{sen}(n\theta)\right]$$

Em particular, se $r = 1$, então:

$$\left[\cos\theta + i \operatorname{sen}\theta\right]^n = \left[\cos(n\theta) + i \operatorname{sen}(n\theta)\right] \quad (1)$$

Façamos: $x = \cos\theta$ e $y = i \operatorname{sen}\theta$ e apliquemos no Teo. binomial:

$$(\cos\theta + i \operatorname{sen}\theta)^n = \cos^n(\theta) + n \cos^{n-1}(\theta)(i \operatorname{sen}\theta) + \frac{n(n-1)}{2!}\cos^{n-2}\theta\,(i \sin\theta)^2 + \cdots$$

$$= \cos^n\theta + i\,n \cos^{n-1}(\theta)\operatorname{sen}\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta \sin^2\theta + \cdots \quad (2)$$

Comparando as equações (1) e (2):

$$\cos(n\theta) + i \operatorname{sen}(n\theta) = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta \sin^2\theta + \cdots + i\,n \cos^{n-1}(\theta)\operatorname{sen}\theta + \cdots$$

Então:

$$\cos(n\theta) = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta \sin^2\theta \pm \cdots$$

$\boxed{18\text{-}10}$

a) $e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}$ sendo: $e^{i\theta} = \cos\theta + i \operatorname{sen}\theta$

$e^{i\phi} = \cos\phi + i \sin\phi$

$$e^{i(\theta+\phi)} = (\cos\theta + i \operatorname{sen}\theta)(\cos\phi + i \operatorname{sen}\phi)$$

$$= \cos\theta\cos\phi + i\cos\theta \operatorname{sen}\phi + i \operatorname{sen}\theta\cos\phi - \operatorname{sen}\theta \operatorname{sen}\phi$$

$$= \cos\theta\cos\phi - \operatorname{sen}\theta \operatorname{sen}\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$$

$$\cos(\theta+\phi) + i \operatorname{sen}(\theta+\phi) = \cos\theta\cos\phi - \operatorname{sen}\theta \operatorname{sen}\phi + i(\operatorname{sen}\theta\cos\phi + \cos\theta\sin\phi)$$

Então: $\cos(\theta+\phi) = \cos\theta\cos\phi - \operatorname{sen}\theta \operatorname{sen}\phi$; $\operatorname{sen}(\theta+\phi) = \operatorname{sen}\theta\cos\phi + \cos\theta$ [illegible] b)

$$A e^{i\theta} \cdot B e^{i\phi} = A B e^{i(\theta+\phi)}$$

A interpretação geométrica do produto de dois números complexos é a seguinte: ampliação e rotação no sentido positivo (anti-horário).

O módulo do vector resultante é o produto dos módulos dos vectores multiplicados;

O ângulo do vector resultante é a soma dos argumentos dos vectores multiplicados.

$\boxed{18.11}$

a) $$\frac{11^{0,5}}{11^{x}} = \frac{3,3166}{2} = 1,6583$$

$$11^{(0,5-x)} = 1,6583 \Rightarrow (0,5-x)\log 11 = \log 1,6583$$

$$\underline{x = 0,289} \Rightarrow \boxed{\log_{11} 2 = 0,289}$$

b) $$\frac{11}{11^{x}} = \frac{11}{7} \Rightarrow 11^{(1-x)} = \frac{11}{7}$$

$$(1-x)\log 11 = \log 11 - \log 7$$

$$1 - x = 1 - \frac{\log 7}{\log 11} \Rightarrow x = \frac{\log 7}{\log 11} = \underline{0,811}$$

$$\boxed{\log_{11} 7 = 0,811}$$