Geometric And Functional Analysis
Vol. 4, No. 5 (1994)
1016-443X/94/0500586-35$1.50+0.20/0

GENERALIZED FLUID FLOWS, THEIR APPROXIMATION AND APPLICATIONS

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Introduction

Consider the motion of an ideal incompressible fluid in a domain $G \subset \mathbf{R}^\nu$. Each two positions of the fluid differ by some permutation of fluid particles. In "classical" hydrodynamics these permutations are assumed to be smooth volume preserving diffeomorphisms of G (see [EM]). If we fix some initial position of each fluid particle we may identify all other positions with corresponding diffeomorphisms, and the configuration space of the fluid is thus identified with the group $\mathcal{D}(G) = \mathcal{D}$ of all such diffeomorphisms.

The fluid flow ξ_t $(t_1 \le t \le t_2)$ is a parametrized path in $\mathcal{D}$, i.e. a family of fluid configurations $\xi_t \in \mathcal{D}$, depending on the parameter t (time). For each flow ξ_t we may define two closely related functionals, the *action* $J\{\xi_t\}_{t_1}^{t_2}$ and the *length* $L\{\xi_t\}_{t_1}^{t_2}$:

$$J\{\xi\}_{t_1}^{t_2} = \int_{t_1}^{t_2} dt \cdot \int_G \frac{1}{2} \left| \frac{\partial \xi_t(x)}{\partial t} \right|^2 dx \ ,$$

$$L\{\xi_t\}_{t_1}^{t_2} = \int_{t_1}^{t_2} dt \cdot \left(\int_G \left| \frac{\partial \xi_t(x)}{\partial t} \right|^2 dx \right)^{1/2} .$$

It is easy to see that

$$J\{\xi_t\}_{t_1}^{t_2} \ge \left(L\{\xi_t\}_{t_1}^{t_2}\right)^2 / 2(t_2 - t_1) \ ,$$

and the equality is achieved if and only if $\int_G \left|\frac{\partial \xi_t(x)}{\partial t}\right|^2 dx = \text{const}$.

The group $\mathcal{D}$ may be naturally embedded into the Hilbert space $L^2(G, \mathbf{R}^\nu)$, because $G \subset \mathbf{R}^\nu$; it is easy to observe that $J\{\xi_t\}_{t_1}^{t_2}$ and $L\{\xi_t\}_{t_1}^{t_2}$ are, correspondingly, the action and the length of the trajectory ξ_t is the metrics induced by this embedding.

Fluid flows, in the absence of external forces, are just geodesic trajectories with respect to the metrics introduced. This means that these flows ξ_t,

$t_1 \le t \le t_2$, are critical points of the functional J and L (if the endpoints ξ_{t_1}, ξ_{t_2} are fixed).

Given two fluid positions $\xi_0, \xi_1 \in \mathcal{D}$, we may look for the shortest path connecting them, i.e. a path ξ_t, $0 \le t \le 1$, such that $J\{\xi_t\}_0^1 = \min$ (and, equivalently, $L\{\xi_t\}_0^1 = \min$). *If such a path exists*, it is automatically a solution of the Euler equations; it is tempting to use such a "Dirichlet principle" to construct these solutions.

But it turned out (see [S1]) that if the dimension $\nu \ge 3$, this variational problem does not have a solution for all the pairs $\xi_0, \xi_1 \in \mathcal{D}$. That is, a diffeomorphism $\xi \in \mathcal{D}(K)$ was constructed, where K is a unit ν-dimensional cube, such that it cannot be connected with the identity diffeomorphism Id by the shortest path in $\mathcal{D}$.

In order to overcome this difficulty, Y. Brenier ([Br1]) introduced the notion of Generalized Flow (GF). This is a wider class of objects (including the smooth flows), where the over-described variational problem always has a solution in this class. (It is similar to the "Generalized Curves" of L. Young ([Y]), but has another nature.)

Our definition of GF is a slight modification of the definition given by Y. Brenier.

Let Ω be a set of points (called "fluid particles" below) with a σ-algebra $\mathcal{B}$ of subsets, and a non-negative finite measure P on $\mathcal{B}$. The Generalized Flow (GF) in G is a measurable mapping

$$x : \Omega \times [t_1, t_2] \to G , \qquad (\omega, t) \to x(\omega, t) .$$

We shall restrict ourselves to the case when $x(\omega, t)$ is continuous in t for almost all $\omega \in \Omega$. So, we have a mapping $\Omega \to X = C(t_1, t_2; G)$, $\omega \to x(\omega, t)$. The image of the measure P in Ω is a measure μ in X called the *distribution* of the GF. In particular, Ω may coincide with X and then $P = \mu$. We may assume that all GF have X as a space of liquid particles, for all observable properties of the flow depend only on its distribution μ. But sometimes it is more convenient to distinguish between the GF and its distribution.

We say, that the GF is *incompressible*, if for each domain $G' \subset G$, for all $t_0 \in [t_1, t_2]$,

$$P\{x(t_0) \in G'\} = \operatorname{mes} G.$$

GF is called "GF with finite action", if

$$J\{P\}_{t_1}^{t_2} = \int_\Omega J\{x(t,\omega)\}_{t_1}^{t_2} P(d\omega) < \infty ,$$

where

$$J\{x(t,\omega)\}_{t_1}^{t_2} = \int_{t_1}^{t_2} \tfrac{1}{2}\left|\dot{x}(t,\omega)\right|^2 dt$$

is the action of a single fluid particle.

We say, that the GF connects Id and $\xi \in \mathcal{D}$, if

$$P\{\omega \mid x(t_1,\omega) \in G_1, x(t_2,\omega) \in G_2\} = \operatorname{mes}\left(G_2 \cap \xi(G_1)\right)$$

for each measurable set $G_1, G_2 \subset G$. In other words, for P-almost all ω,

$$x(t_2,\omega) = \xi\left(x(t_1,\omega)\right) .$$

We see that the incompressible GFs with finite action, connecting Id and ξ, form a (very special) class of random processes in G. In most cases we may identify GF and its distribution μ.

Y. Brenier proved in [Br1], that for each $\xi \in \mathcal{D}$, there exists an incompressible GF μ in G, connecting Id and ξ, and such that $J\{\mu\} = \min$, where min is looked for among all incompressible GFs, connecting Id and ξ.

The nature of these generalized minimal flows is still unclear. Examples, presented by Brenier, display unusual behavior of these GF. The delicate question is, to what extent may these minimal GF be regarded as generalized solutions of the Euler equations. (The work [Br2] shows that the similarity is very close, and even that for minimal GF there exists a function $p(x,t)$ playing the role of pressure.)

This work concerns other aspects of Generalized Flows. They proved to be a powerful and flexible tool for investigating the structure of the space $\mathcal{D}$. The theorem of approximation of GF by smooth flows plays a key role.

Let $\xi_t \subset \mathcal{D}$ be a smooth flow. It may be regarded as GF with the space of the fluid particles $\Omega = G$. Let us denote by μ_{ξ_t} the distribution of this GF. This is a measure in $X = C\left([0,1], G\right)$, and we may define a new GF with $\Omega = X$, $\mathcal{B} =$ Borel Σ-algebra in X, and μ_{ξ_t} the measure in X; the mapping $\Omega \times [0,1] \to G$ is defined in a trivial way: $\left(x(\cdot), t\right) \to x(t)$. This GF (denote it once more μ_{ξ_t} or $\mu_{\xi_t}\{dx\}$) may be called a *standard representation* of GF ξ_t.

In what follows we confine ourselves to the case when

$$G = K = \left\{x = (x_1, \ldots, x_\nu) \in \mathbf{R}^\nu \mid |x_i| < \tfrac{1}{2}\right\}$$

is a unit ν-dimensional cube.

The main result of this work is the following

APPROXIMATION THEOREM. *If the dimension $\nu \geq 3$, then each incompressible* GF $\mu\{dx\}$, *connecting* Id *and* $\xi \in \mathcal{D}$, *may be approximated by the smooth flows* $\xi_{k,t}$, *together with the action. This means that there exists a sequence of smooth flows* $\xi_{k,t}$, *connecting* Id *and* ξ, *such that*

(i) *the measures* $\mu_{\xi_{k,t}}$ *converge weak* $*$ *in* X *to the measure* μ;

(ii) $J\{\xi_{k,t}\}_0^1 \to J\{\mu\}_0^1$.

The first part of the work is devoted to the proof of this theorem. Note that the assertion of the theorem is false for $\nu = 2$, which will be explained in detail below.

The second part of the work is devoted to the applications of the Approximation Theorem. We obtain sharp estimates for the diameter of $\mathcal{D}$ (one of the main results of [S1] is, that if $\nu \geq 3$, then $\operatorname{diam}\mathcal{D} < \infty$; but the estimates for $\operatorname{diam}\mathcal{D}$, obtained there, are very weak).

The next result relates the "Hölder property" of $\mathcal{D}$. It was proved in [S1] that if $\nu \geq 3$, then there exist $C > 0$, $\alpha > 0$, such that for each $\xi, \eta \in \mathcal{D}$,

$$\operatorname{dist}(\xi, \eta) \leq C\|\xi - \eta\|_{L^2}^{\alpha} ,$$

where we regard $\mathcal{D}$ to be embedded in $L^2(K, \mathbf{R}^\nu)$, so that $\xi, \eta \in L^2$, and

$$\operatorname{dist}(\xi, \eta) = \inf_{\substack{\xi_t \subset \mathcal{D} \\ \xi_0 = \xi \\ \xi_1 = \eta}} L\{\xi_t\}_0^1 .$$

The power α was estimated from below, but the lower bound obtained for α was very small (for $\nu = 3$, we found that $\alpha \geq \frac{1}{64}$). In this work, using the Approximation Theorem, we find a better bound: $\alpha \geq \frac{2}{\nu+4}$ $(\nu \geq 3)$. An intriguing question is, what is the best lower estimate? Is it true, that $\alpha > 1 - \varepsilon$ for all $\varepsilon > 0$?

The next application of the GF is the lower estimate for the action of 2-dimensional flow, connecting Id and a mapping ξ, having some twisting property. This estimate implies, that if $\nu = 2$, then $\operatorname{diam}\mathcal{D} = \infty$. The last result is a particular case of the theorem of Eliashberg and Ratiu ([ElR]), asserting that the diameter of the symplectomorphism group is infinite. But our estimate itself is not covered by the results of [ElR].

We use this estimate to construct a non-attainable diffeomorphism in the 2-dimensional case.

It was proved in [S2] that if $\nu \geq 3$, then each $\xi \in \mathcal{D}$ is attainable, i.e. there exists a flow ξ_t, connecting Id and ξ, such that $J\{\xi_t\}_0^1 < \infty$. Here we show that for $\nu = 2$ it is not true.

The next application of GF is a simpler and more transparent proof than in [S1], that if $\nu \geq 3$ there exists $\xi \in \mathcal{D}$ such that $\min J\{\xi_t\}_0^1$ is not achieved among the smooth paths ξ_t, connecting Id and ξ.

The last application of the Approximation Theorem is a simple proof of existence of conjugate points on the geodesics in $\mathcal{D}(G)$, if $\dim G \geq 3$. Recently, G. Misiolek ([Mi]) proved that in the group of area preserving diffeomorphisms of a 2-dimensional torus there exist both geodesics, carrying conjugate points, and geodesics without conjugate points. (It is very likely that this assertion is true for all compact 2-dimensional manifolds and bounded 2-dimensional domains.) If the dimension $\nu \geq 3$, then the conjugate points are indispensible for each sufficiently long geodesic line. However, the nature of conjugate points and the proof of their existence in 2 and 3-dimensional cases are quite different and have in fact nothing in common.

Note that although this work improves the results of [S1], it is based on the technical theorems contained in [S1].

It should be noted, that there exist a number of results concerning the structure of completions of configuration space $\mathcal{D}$. To each diffeomorphism $\xi \in \mathcal{D}$ we may put in correspondence an operator T_ξ in $L^2(\mathcal{D}) : T_\xi u(x) = u(\xi(x))$. Thus we obtain a representation T of the group $\mathcal{D}$ in the group U of unitary operators in $L^2(G)$. Let $T_\mathcal{D}$ be the image of $\mathcal{D}$. Elements of the closure of this group are "generalized configurations" of the fluid.

Closure of $T_\mathcal{D}$ in the norm topology coincides with $T_\mathcal{D}$, because $\|T_\xi - T_\eta\| = 2$, if $\xi \neq \eta$.

Closure of $T_\mathcal{D}$ in the *strong operator* topology is the semigroup $S\,\mathrm{mes}(G)$ of operators $T_f : u(x) \to u(f(x))$, where $f : G \to G$ is a measurable and measure preserving mapping (i.e. $\mathrm{mes}\, f^{-1}(A) = \mathrm{mes}\, A$ for each measurable set $A \subset G$). This was proved in [BFR], [A], [Mo2].

Instead of the strong topology, we may introduce a distance dist (see above) and ask, what is the *completion* $\overline{\mathcal{D}}$ of $\mathcal{D}$ as a metric space with this distance. It was proved in [S1], that if the dimension $\nu \geq 3$, then this *completion* coincides with $S\,\mathrm{mes} G$, and if $\nu = 2$, then $\overline{\mathcal{D}}$ and $\overline{S\,\mathrm{mes} G}$ are different (in fact, the nature of $\overline{\mathcal{D}}$, if $\nu = 2$, is still unclear).

Next, if we consider the closure of $T_\mathcal{D}$ in the *weak operator* topology, then we shall arrive at the semigroup of *bistochastic operators*, or, in the terminology of A.M. Vershik, *polimorphisms*. These are operators in $L^2(G)$ of the form $Ku(x) = \int_G K(x,y)u(y)du$, where the Kernel $K(x,y)$ possesses the following properties:

(i) $K(x,y) \geq 0$ (i.e. K is a positive measure in $G \times G$);

(ii) $\displaystyle\int_G K(x,y)dx \equiv 1$;

(iii) $\displaystyle\int_G K(x,y)dy \equiv 1$.

See [V], for a detailed description of the operators. Bistochastic operators were used by Y. Brenier in his works [Br1], [Br2], about the existence of minimizing generalized flows. They were the main tool in the work [S3], concerning existence and properties of the stationary flows in 2-dimensional domains.

Embedding of an infinite-dimensional group into some "enveloping" semigroup is a powerful method in representation theory; see, for example [N] and the references there.

Acknowledgements.

Most of this work was done during my stay at Ecole Normale Supérieure in Paris in February, 1993. I am very thankful to this institution for giving me the opportunity to come and take advantage of the excellent working conditions. Personally, I am thankful to Yann Brenier for many fruitful discussions.

I am thankful to the referee for his useful remarks.

1. Proof of the Approximation Theorem

Given a GF μ, connecting Id and ξ, we shall construct a sequence $\xi_{k,t}$ of smooth flows, connecting Id and ξ, and such that $\mu_{\xi_{k,t}} \to \mu$ weak $*$, and $J\{\xi_{k,t}\}_0^1 \to J\{\mu\}_0^1$. Our construction consists of a number of steps. On each step we introduce a new type of GF, and prove that these GF may approximate each GF introduced in the previous step. On the last step the smooth flows appear. It is evident that this implies that the smooth flows approximate the generic GF.

Step 1. Given $\varepsilon > 0$, let us define the following transformation f_ε of the space $C(0,1;K)$:

$$x(t) \to f_\varepsilon x(t) = \begin{cases} x\big(t/(1-\varepsilon)\big)\ , & 0 \le t \le 1-\varepsilon \\ x(1)\ , & 1-\varepsilon \le t \le 1\ . \end{cases}$$

It is clear that f_ε is measurable and transforms each GF μ into some other GF $f_\varepsilon(\mu)$.

LEMMA 1.1. *If* $\varepsilon \to 0$, *then* $f_\varepsilon\mu \rightharpoonup \mu$ *weak* $*$ *and* $J\{f_\varepsilon\mu\} \to J\{\mu\}$.

Proof: For each $\delta>0$ there is a constant $A>0$ such that $f_\varepsilon\mu\{x \big| \|x\|_{H^1(0,1;K)} > A\} < \delta$, for all ε, $0 \le \varepsilon \le \frac{1}{2}$. The set $\beta_A = \{x \in \Omega \mid \|x\|_{H^1} \le A\}$ is compact in Ω. Using the Stone-Weierstrass theorem, we see that the functionals of the form

$$\phi\{x(t)\} = \phi_{t_1,\dots,t_N}\big(x(t_1),\dots,x(t_N)\big) \qquad (1.1)$$

are dense in $C(\beta_A)$. Hence, it is sufficient to prove that

$$\int_\Omega \phi\{x(t)\} f_\varepsilon \mu\{dx\} \to \int_\Omega \phi\{x(t)\}\mu\{dx\}$$

for each bounded functional $\phi\{x\}$ having the form (1.1). But

$$\int_\Omega \phi\{x(t)\} f_\varepsilon \mu\{dx\} = \int_{\beta_A} + \int_{\Omega\setminus\beta_A} ;$$

the second integral is less than δ, while the first tends to $\int_{\beta_A} \phi\{x\}\mu\{dx\}$ because of the equicontinuity of the family β_A.

The second assertion of the lemma follows from the equality

$$J\{f_\varepsilon \mu\} = (1-\varepsilon)^{-1} J\{\mu\} . \qquad \square$$

So, we may confine ourselves to GF μ, such that for μ – almost all trajectories $x(t)$, $x(t) = \text{const}$ for $1-\varepsilon \le t \le 1$.

Likewise, we may approximate the GF μ by GF μ', such that it is immobile near ∂K; this means that if $x(0)$ is in the ε-neighbourhood of ∂K, then $x(t) \equiv x(0)$.

Step 2. Given a GF μ, let us choose trajectories $X_1(t), X_2(t), \ldots$ independently, with the same probability distribution $\mu\{dx\}$. The measures

$$\mu_p\{dx\} = \sum_{k=1}^{p} \frac{1}{p}\delta\{x - x_k\}$$

converge weakly to the measure μ with probability 1. This implies that these GF are asymptotically incompressible with probability 1.

Suppose that this is true for our sequence $x_1(t), x_2(t), \ldots$. Moreover, suppose that

$$J\{\mu_p\} \to J\{\mu\} \qquad (p \to \infty) ;$$

this is also true with probability 1.

Thus we have approximated our measure μ by the measures μ_p, concentrated on a finite number of trajectories.

Step 3. Now we shall approximate the GF μ by *smooth multiflows.* We shall give the Eulerian and Lagrangian descriptions of a multiflow.

DEFINITION 1.1 (Eulerian definition of a multiflow). *A smooth multiflow of order N is a collection of smooth functions $\rho_1(x,t),\dots,\rho_N(x,t)$ and smooth vector fields $v_1(x,t),\dots,v_N(x,t)$, $x \in K$, $0 \le t \le 1$, such that*

(i) $\sum_{i=1}^N \rho_i(x,t) \equiv 1$;

(ii) $\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i v_i) \equiv 0$;

(iii) *the fields v_i are tangent to ∂K.*

To define the *Lagrangian multiflow*, consider N copies of $K_1,\dots,K_N$ of K; let $a_i(x)$ be smooth non-negative functions in K_i (density functions). Consider N time-dependent diffeomorphisms $\eta_{i,t} : K_i \to K$ $(i = 1,\dots,N)$, $0 \le t \le 1$.

DEFINITION 1.2 (Lagrangian definition of a multiflow). *A collection $\mathcal{M} = \{a_1,\dots,a_N;\eta_{1,t},\dots,\eta_{N,t}\}$ is called a (Lagrangian) multiflow, if*

(i)
$$\sum_{i=1}^N \rho_i(x,t) \equiv 1 \quad \text{in} \quad K,$$

where
$$\rho_i(x,t) = a\big(\eta_t^{-1}(x)\big) \cdot \left|\frac{\partial \eta_{i,t}^{-1}}{\partial x}\right|$$

is the density of the i-th phase;

(ii) *for each $i \le N$, and each $x_i \in K_i$,*
$$\eta_{i,1}(x_i) = \xi \circ \eta_{i,0}(x_i) \ ,$$

where $\xi \in \mathcal{D}(K)$ is a given diffeomorphism.

(ii) means that the multiflow $\mathcal{M}$ connects Id and ξ.

Given a multiflow $\mathcal{M}$, we may define corresponding GF $\mu_{\mathcal{M}}$ in the following way: if $\mathcal{M} = \{a_1,\dots,a_N;\eta_{1,t},\dots,\eta_{N,t}\}$, $\phi \in C(\Omega)$, then

$$\int_\Omega \phi\{x\}\mu_{\mathcal{M}}\{dx\} \stackrel{\text{def}}{=} \sum_{i=1}^N \int_K a_i(x_i)\phi\{\eta_{i,t}(x_i)\}dx_i \ .$$

Given a GF μ, we shall approximate it by a multiflow. This means that we construct the sequence of multiflows

$$\mathcal{M}^{(k)} = \{a_1^k,\dots,a_{p_k}^k;\eta_{1,t}^k,\dots,\eta_{p_k,t}^k\} \ ,$$

such that for each function $\phi(x_1,\dots,x_N) \in C(K^N)$ and for each sequence $0 \le t_1 < t_2 < \dots < t_N \le 1$,

$$\int \phi\big(x(t_1),\dots,x(t_N)\big)\mu_{\mathcal{M}^{(k)}}\{dx\} \to \int \phi\big(x(t_1),\dots,x(t_N)\big)\mu\{dx\} \ .$$

(This is sufficient for the weak $*$ convergence.)

Let us choose the trajectories $x_1(t),\dots,x_k(t)$ as described in Step 2. Let $\varphi(x) = \varphi(|x|) \in C_0^\infty(\mathbf{R}^\nu)$, $\varphi(x) \ge 0$, $\varphi(x) = 0$ for $|x| > 1$, and $\int \varphi(x)dx = 1$. We may define a sequence of multiflows $\overline{\mathcal{M}}^k = (a_1^k,\dots,a_k^k;\eta_{1,t}^k,\dots$ where

$$a_i^k(x) = \frac{1}{k\varepsilon^\nu}\varphi\left(\frac{x - x_i(0)}{\varepsilon}\right) \ ,$$

if $\operatorname{dist}(x_i(0),\partial K) > \varepsilon$, and

$$a_i^k(x) = \frac{1}{k\varepsilon^\nu}\sum_{\gamma\in\Gamma}\varphi\left(\frac{x - \gamma x_i(0)}{\varepsilon}\right) \ ,$$

if $\operatorname{dist}(x_i(0),\partial K) \le \varepsilon$, where Γ is the discrete group of motions in $\mathbf{R}^\nu$, generated by the reflections in the faces of K. The mappings $\eta_{i,t}^k$ are of the form $\eta_{i,t}^k(x) = x_i(t) + (x - x_i(0))$. Here we assume that ε is sufficiently small, so that the initial GF μ (obtained in Step 1) is fixed in the 2ε-neighbourhood of ∂K.

The multiflow $\overline{\mathcal{M}}^k$ is not volume-preserving, and it does not, in general, satisfy the boundary condition $\eta_{i,1}(x) = \xi\cdot\eta_{i,0}(x)$. To improve it, let us choose a flow $\xi_t \subset \mathcal{D}$, $0 \le t \le 1$, $\xi_0 = \mathrm{Id}$, $\xi_1 = \xi$, and put

$$\eta_{i,t}^k(x) = x_i(t) + \xi_t(x) - \xi_t\big(x_i(0)\big) \ .$$

This choice of mappings $\eta_{i,t}^k$ ensures the boundary conditions $\eta_{i,1}^k = \xi\circ\eta_{i,0}^k$. But it is still not quite incompressible. We improve it, multiplying by a suitable mapping $\zeta_t^k : K \to K$. Let $\rho^k(x,t) = \sum_{i=1}^k a_i(\eta_{i,t}^{k^{-1}}(x))$.

LEMMA 1.2. *If $k \to \infty$, then with probability* 1

(1) $\sup_{x,t}|\rho^k(x,t) - 1| \to 0$,

(2) $\sup_{x,t}|\partial_x^\alpha\rho^k(x,t)| \to 0$, $(|\alpha| > 0)$

(3) $\int_K\int_0^1 |\partial_t\rho^k(x,t)|^2 dx\,dt \to 0$.

Proof: Note that $E\rho^k(x,t) \equiv 1$, where E is the mean value with respect to the probability distribution $\underbrace{\mu \otimes \ldots \otimes \mu}_{k \text{ times}}$. From the definition of ρ^k we see that $|\partial_x^\alpha \rho^k| \leq C_\alpha$ for all α. From the Large Numbers Law we see that for every point (x,t), $\mathrm{Prob}\{\partial_x^\alpha \rho(x,t) \to 0\} = 1$. But $|\partial_x^\alpha \rho| \leq C_\alpha$ for all α, $|\alpha| > 0$.

Moreover,

$$E\left|\frac{\partial}{\partial t}\partial_x^\alpha \rho(x,t)\right| \leq C_\alpha E|\dot{x}(t)|^2 ,$$

and, therefore, we may choose a finite set of points (x_j, t_j), dense enough in $K \times [0,1]$; at each point of this set with probability 1, $\rho \to 1$, and $\partial_x^\alpha \rho \to 0$. If $|\rho(x,t_2) - \rho(x,t_1)| > a$, the action $J > c \cdot a\varepsilon^\nu \cdot \frac{\varepsilon^2}{|t_2 - t_1|}$; hence, $a < cJ|t_2 - t_1|\varepsilon^{-\nu-2}$. This means that if we take the time moments $0 \leq t_1 < \ldots < t_M \leq 1$ such that $t_{i+1} - t_i < caJ^{-1}\varepsilon^{\nu+2}$, and a finite number of points (x_j, t_i), such that the distance between each point x and the closest point x_j is less than $c \cdot a \cdot \varepsilon^{-\nu-1}$, and the sequence of the paths $x_i(t)$ is typical, i.e. $\lim_{N\to\infty} \frac{1}{N}\sum_{p=1}^N \rho_p(x_j, t_i) = 1$, then $|\rho(x,t) - 1| < a$ for all x, t. A similar reasoning proves (2).

To prove (3), note that $u_j(x,t) = \frac{\partial \rho_j(x,t)}{\partial t}$ are L^2-functions, chosen independently and with the same distribution, $E\|u_j\|^2 < \infty$, $Eu_j = 0$. If $u^{(k)} = \frac{\partial \rho^k}{\partial t} = \frac{1}{k}\sum_{j=1}^k u_j$, then

$$\begin{aligned} E\|u^{(k)}\|^2 = E\left\|\frac{1}{k}\sum_{j=1}^k u_j\right\|^2 &= \\ &= \frac{1}{k^2}\sum_{j=1}^k E\|u_j\|^2 + \frac{1}{k^2}\sum_{i\neq j} E(u_i, u_j) = \\ &= \frac{1}{k}E\|i_j\|^2 , \end{aligned}$$

for u_i, u_j are *independent*, and hence $E(u_i, u_j) = (u_i, Eu_j) = 0$. By the Large Numbers Law, $\|u^{(k)}\|^2 \xrightarrow[k\to\infty]{} 0$ with probability 1. This proves (3).

Now we must construct a correcting flow ζ_t^k, such that the multiflow $\mathcal{M} = \{a_1, \ldots, a_k; \zeta_t^k \circ \eta_{1,t}^k, \ldots, \zeta_t^k \circ \eta_{k,t}^k\}$ is incompressible. This means that $\left|\frac{\partial \zeta_t^k(x)}{\partial x}\right| = \rho(x,t)$, where $\rho = \rho^k$. We have just proved that ρ is close to 1 with all derivatives, if k is large enough. This is a problem solved in the most general case by J. Moser ([ElR]), but here we use a much simpler and more explicit approach. Let us describe it for the case $\nu = 2$. Let us choose a function $\gamma(x) \in C_0^\infty[-\frac{1}{2}, \frac{1}{2}]$, $\gamma(x) \geq 0$. Let $g(y, s; x) = y + s \cdot \gamma(x)$. Let $\rho(x)$ be a density in the square $K^2 = \{ -\frac{1}{2} < x_i < \frac{1}{2} \}$ close enough to 1 with its derivatives up to order d.

To construct the mapping ζ, we choose the function $s(x_2)$, so that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\tfrac{1}{2} + g\big(x_2, s(x_2), x_1\big)\Big) dx_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 \int_{-\frac{1}{2}}^{x_2} \rho(x_1, x_2') dx_2' \ ;$$

this relation determines $s(x_2)$ uniquely, and $s(x_2)$ is close to 0 (with derivatives), if $\rho(x_1, x_2)$ is close to 1. Now, let us define the mapping ζ in the form $\zeta(x_1, x_2) = (y_1, y_2)$; $y_2 = g(x_2, s(x_2), y_1)$;

$$\int_{-\frac{1}{2}}^{x_1} \rho(x, x_2) dx = \int_{-\frac{1}{2}}^{y_1} \left(1 + \gamma(x) \frac{\partial s}{\partial x_2}\right) dx \ .$$

This relation just means that the mapping ζ_t transforms the measure $\rho\, dx_1\, dx_2$ into $dy_1\, dy_2$. It is easy to see that

$$\sup_{x \in K} \left| \frac{\partial \eta_t}{\partial x} - \mathrm{Id} \right| \leq C \sup \left| \frac{\partial \rho}{\partial x} \right| \ ,$$

$$\int_K \left| \frac{\partial \eta_t}{\partial t} \right|^2 dx \leq C \sup(\rho^{-1}) \cdot \int_K \left| \frac{\partial \rho}{\partial t} \right|^2 dx \ ,$$

and by Lemma 1.2, with probability 1

$$J\{a_1, \ldots, a_k; \zeta_t^k \circ \eta_{1,t}^k, \ldots, \zeta_t^k \circ \eta_{k,t}^k\} \to J\{\mu\} \ .$$

Thus, we constructed the sequence of incompressible multiflows $\mathcal{M}^k$, connecting Id and ξ, such that $\mu_{\mathcal{M}^k} \xrightarrow{w} \mu$, and $J\{\mathcal{M}^k\}_0^1 \to J\{\mu\}_0^1$.

Step 4. Now we shall approximate a multiflow $\mathcal{M}$ by smooth flows. This in turn requires a number of steps.

Given a multiflow $\mathcal{M}$, we shall construct a sequence of smooth flows $\sigma_{i,t} \subset \mathcal{D}$, $\sigma_{i,0} = \mathrm{Id}$, $\sigma_{i,1} = \xi$, such that $\mu_{\sigma_{i,t}} \to \mu_{\mathcal{M}}$ weak $*$.

Let us fix some multiflow $\mathcal{M} = (a_1(x), \ldots, a_p(x); \xi_{1,t}(x), \ldots, \xi_{p,t}(x))$; $a_i \in C^\infty(K_i)$, $\xi_{i,t} : K_i \to K$, and $\sum_{i=1}^p a_i(\xi_{i,t}^{-1}(y)) |\partial \xi^{-1}(y)/\partial y| \equiv 1$.

Step 4.1. We may assume that $a_p(x,t) \equiv \frac{1}{p}$; otherwise we may construct a new multiflow

$$\mathcal{M}' = \{a_1', \ldots, a_{p+1}'; \xi_{1,t}', \ldots, \xi_{p+1,t}'\} \ ,$$

where $a_i'(x) = \frac{p}{p+1} a_i(x)$, $\xi_{i,t}' = \xi_{i,t}$ $(1 \leq i \leq p)$, $a_{p+1}'(x) \equiv \frac{1}{p+1}$, $\xi_{p+1,t}' = \xi_t$, where $\xi_t \subset \mathcal{D}$ is some fixed flow, connecting Id and ξ.

If $p \to \infty$, $\mathcal{M}'$ is asymptotically weakly close to $\mathcal{M}$.

Step 4.2. On this step, we shall represent a multiflow $\mathcal{M}$ as a composition of 2-flows (i.e. multiflows with $p = 2$).

DEFINITION: Let

$$\mathcal{M} = (a_1, \dots, a_p; \xi_{1,t}, \dots, \xi_{p,t}) ,$$
$$\mathcal{M}' = (b_1, \dots, b_q; \eta_{1,t}, \dots, \eta_{q,t})$$

be two multiflows such that $a_p(x) \equiv a_p = \text{const} > 0$. Then the multiflow $\mathcal{M}'' = \mathcal{M} * \mathcal{M}'$, the composition of $\mathcal{M}$ and $\mathcal{M}'$ is defined as follows:

$$\mathcal{M}'' = (c_1, \dots, c_{p+q-1}; \zeta_{1,t}, \dots, \zeta_{p+q-1,t}) ,$$

where

$$\begin{aligned}
&c_1 = a_1, \dots, c_{p-1} = a_{p-1} \\
&c_p = a_p \cdot b_1, c_{p+1} = a_p \cdot b_2, \dots, c_{p+q-1} = a_p \cdot b_q ; \\
&\zeta_{1,t} = \xi_{1,t}, \dots, \zeta_{p-1,t} = \xi_{p-1,t} ; \\
&\zeta_{p,t} = \xi_{p,t} \circ \eta_{1,t}, \dots, \zeta_{p+q-1,t} = \xi_{p,t} \circ \eta_{q,t}
\end{aligned}$$

Our next step is representing the multiflow $\mathcal{M} = (a_1, \dots, a_p; \xi_1^t, \dots, \xi_p^t)$, where $a_p \equiv \frac{1}{p}$, as a composition of $(p-1)$ 2-flows $\mathcal{M}^i = (a^i, b^i; \eta_{i,t}, \zeta_{i,t})$, where $b^i(x) \equiv b_i = \frac{i}{i+1}$.

So we are looking for 2-flows, $\mathcal{M}^i$, such that

$$\mathcal{M} = \mathcal{M}^1 * (\mathcal{M}^2 * (\dots * \mathcal{M}^{p-1}) \dots) ;$$

This means that

$$\begin{aligned}
&a_1(x) = a^1(x) , \\
&a_2(x) = b^1(x) a^2(x) = b_1 a^2(x) , \\
&a_3(x) = b_1 b_2 a^3(x) , \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_p(x) = b_1 b_2 \dots b_{p-1} ,
\end{aligned}$$

(where $b_i = \frac{i}{i+1}$)

$$\begin{aligned}
&\xi_{1,t} = \eta_t^1 , \\
&\xi_{2,t} = \zeta_t^1 \circ \eta_t^2 , \\
&\xi_{3,t} = \zeta_t^1 \circ \zeta_t^2 \circ \eta_t^3 , \\
&\dots\dots\dots\dots\dots\dots \\
&\xi_{p,t} = \zeta_t^1 \circ \dots \circ \zeta_t^{p-1} .
\end{aligned}$$

The densities $a^i(x) = \frac{a_i(x)}{b_1 \cdot \dots \cdot b_{i-1}}$. In order to find η_t^i, ζ_t^i, let us first set $\eta_t^1 \equiv \xi_{1,t}$. Then we find a flow ζ_t^1 such that $\mathcal{M}^1$ is an incompressible 2-flow, i.e. $\zeta_t^{1*}(b_1 dx) + \eta_t^{1^*}(a_1 dx) = dx$. This is a problem of Moser type. After solving it (for example, in an explicit way, as above, in Lemma 1.2), we obtain a flow ζ_t^1, smooth in t. Now, set $\eta_t^2 = (\zeta_t^1)^{-1} \circ \xi_{2,t}$, and find ζ_t^2, satisfying relation $\zeta_t^{2*}(b_2 dx) + \eta_t^{2*}(a_2 dx) = dx$, once more using Lemma 1.2; etc.

After $(p-1)$ steps we shall obtain the desired representation of a multiflow $\mathcal{M}$ as a composition of 2-flows $\mathcal{M}^i$, $i = 1, \dots, p-1$. All 2-flows $\mathcal{M}^i$ are smooth in x, t, if $\mathcal{M}$ is, and satisfy the boundary conditions

$$\begin{aligned} \eta_1^1 &= \xi \circ \eta_0^1 \ ; \\ \zeta_1^1 &= \xi \circ \zeta_0^1 \ ; \\ \eta_1^i &= \eta_0^i \ , \\ \zeta_1^i &= \zeta_0^i \quad (i = 2, \dots, p-1) \ ; \end{aligned}$$

Step 5. Given an incompressible 2-flow $\mathcal{M} = (a(x), b(x); \eta_t, \zeta_t)$, such that $\eta_1 = \xi \circ \eta_0$, $\zeta_1 = \xi \circ \zeta_0$, we shall construct a smooth flow σ_t, approximating it in a weak sense together with the action and such that $\sigma_0 = \mathrm{Id}$, $\sigma_1 = \xi$.

Suppose that $b(x) \equiv b = \text{const}$, $a(x) \in C_0^\infty(K)$, $\int_K a(x)dx = 1 - b$, and $\eta_t^{*^{-1}}(a\,dx) + \zeta_t^{*^{-1}}(b\,dx) \equiv dx$. Let $\rho(x,t)dx = \eta_t^{*^{-1}}(a\,dx)$, $(1 - \rho(x,t))dx = \zeta_t^{*^{-1}}(b\,dx)$.

These are the densities of the 1st and 2nd phases. Let $v(x,t) = \dot\eta_t \circ \eta_t^{-1}(x)$, $w(x,t) = \dot\zeta_t \circ \zeta_t^{-1}(x)$ be the velocity fields of the phases, so that $\operatorname{div}(\rho v) + \operatorname{div}(1-\rho)w = 0$, and $\rho \equiv 0$ in some neighbourhood of ∂K.

The first substep is the construction of a discontinuous flow in K, approximating 2-flow $\mathcal{M}$ together with the action.

Step 5.1. Let $u(x,t) = \rho v(x,t) + (1-\rho)w(x,t)$ be the mean velocity field. It is incompressible and vanishes in the neighbourhood of ∂K. Let $\widetilde{\xi}_t$ be the flow generated by the field $u(x,t)$, $\widetilde{\xi}_0 = \mathrm{Id}$. Then for the 2-flow $\mathcal{M}' = (a, b; \widetilde{\xi}_t^{-1} \circ \eta_t, \widetilde{\xi}_t^{-1} \circ \zeta_t)$ the mean velocity field $u \equiv 0$; if we approximate $\mathcal{M}'$ by the smooth flow σ_t', the flow $\widetilde{\xi}_t \circ \sigma_t' = \sigma_t$ will approximate 2-flow $\mathcal{M}$. So it is sufficient to consider the 2-flows with the mean velocity equal to 0. Thus we have excluded the mean motion. Explicitly this means that $\rho v + (1-p)w \equiv 0$; $w = -\frac{\rho}{1-\rho} v$.

Step 5.2. Let us divide the time interval $T : 0 \le t \le 1$ into N equal subintervals $T_i : \frac{i-1}{N} \le t \le \frac{i}{N}$ $(i = 1, \dots, N)$. Consider a modified vector field $v^N(x,t) = v\big(x, \frac{i-1}{N}\big)$, if $\frac{i-1}{N} \le t < \frac{i}{N}$. This is a piecewise smooth vector field, discontinuous in t. Let $\rho^N(x,t)$ be the solution of the mass conservation equation

$$\frac{\partial \rho^N}{\partial t} + \operatorname{div}(\rho^N v^N) = 0$$

$$\rho^N(x,0) = \rho(x,0) \ .$$

It is easy to see that $\rho^N(x,t) \to \rho(x,t)$ uniformly together with all derivatives in x, when $N \to \infty$.

Let $w^N = -\frac{\rho^N}{1-\rho^N} v^N$; let η_t^N, ζ_t^N be the flows satisfying the equations

$$\frac{\partial \eta_t^N(x)}{\partial t} = v_N(t, \eta_t^N(x)) \ , \quad \frac{\partial \zeta_t^N(x)}{\partial t} = w_N(t, \zeta_t^N(x))$$

with the initial conditions $\eta_0^N(x) \equiv \eta_0(x)$, $\zeta_0^N(x) \equiv \zeta(x)$. Then $\mathcal{M}^N = (a, b; \eta_t^N, \zeta_t^N)$ is an incompressible 2-flow, and $\mathcal{M}^N$ approximates $\mathcal{M}$ weakly, when $N \to \infty$.

Step 5.3. Now we introduce the notion of a sand-like domain.

Let $C > 2$, $1 > r > 0$, $H > 1$ be fixed (in what follows, r will be small and H large). Consider a cubic lattice $\mathcal{L}_1$ in K with mesh size $C \cdot r$; let $\mathcal{U}_1$ be the union of the balls of radius $r_1 = r$ with centers at the points of the lattice $\mathcal{L}_1$, contained entirely in K. These (disjoint) balls we shall call *the balls of the first generation.*

Let $\mathcal{L}_2$ be the cubic lattice with mesh size $C \cdot H^{-1}$; denote by $\mathcal{U}_2$ the union of the balls with the centers in the points of $\mathcal{L}_2$, having radius $r_2 = r \cdot H^{-1}$, contained entirely in K and such that the distance between each ball of $\mathcal{U}_2$ and each ball of $\mathcal{U}_1$ is more than r_2 (these balls are called *the balls of the 2nd generation*). Proceeding in the same manner, we obtain the domains $\mathcal{U}_3, \dots, \mathcal{U}_n$, consisting of the balls of 3rd, $\dots$, n-th generations.

Let $\mathcal{U} = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_n$. Its volume is more than $1 - (1 - \alpha + \varepsilon)^{n-1}$, where $\alpha = (Cr)^{-\nu} \cdot$ (volume of the ball of radius r) < 1, and $\varepsilon \to 0$, when $H \to \infty$.

Hence, choosing n sufficiently large, we may reach the volume of $\mathcal{U}$ arbitrarily close to 1. $\mathcal{U}$ may be called a sand-like domain.

Step 5.4. Now, we turn to the approximation of a 2-flow by a discontinuous flow, which is a preliminary step in approximating it by a smooth flow. We shall divide K into 2 domains with piecewise-smooth boundary, $R(t)$ and $S(t) = K \setminus R(t)$. These domains depend on t, and they bear 2 phases of the discontinuous flow, approximating the 2-flow. We are working on the time interval I_i; hence, the vector field $v^N(x,t) = v(x, t_{i-1})$ is constant in time and $w^N(x,t) = -\frac{\rho(x,t)}{1-\rho(x,t)} v^N(x,t)$ has a constant direction. Let us define the domain R at the time moment $t = t_{i-1}$. Let us divide K into 3 sets $K = K_0 \cup K_1 \cup K_2$;

$$\begin{aligned} K_0 &= \{x | v(x, t_{i-1}) = 0\} \ ; \\ K_1 &= \{x \mid 0 < |v(x, t_{i-1})| < \delta\} \ ; \\ K_2 &= K \setminus (K_0 \cup K_1) \ . \end{aligned}$$

If $\operatorname{mes} K_0 = 0$, we set $R_{t_{i-1}} \cap (K_0 \cup K_1) = \emptyset$. Otherwise, we choose some δ_1, $0 < \delta_1 < \delta$, and set $K_0' = \{x \mid |v(x, t_{i-1})| < \delta_1\}$. Then we take $R_{t_{i-1}} \cap (K_1 \setminus \overline{K}_0') = \emptyset$. Let us describe $R_{t_{i-1}} \cap K_0'$. Let $\rho(x, t_{i-1})$ be the density of the 1st phase at the moment t_{i-1}. Density is bounded off from 1: $\rho(x,t) < \rho_0 < 1$. Let us construct a sand-like domain $\mathcal{U}$ in K_0' with parameters r, H, C, so that r is small enough. Then the finite number of generations (say, n) is enough for the average density ρ_1 of $\mathcal{U}$ to be more than ρ_0. Then $\mathcal{U} \cap K_0'$ is a union of a finite number of disjoint balls, $\mathcal{U} \cap K_0' = \bigcup_{i=1}^M B(x_i, r_i)$ with the center at the point x_i and radius r_i. Let $R = \bigcup_{i=1}^M B(x_i, \beta_i r_i)$, where $\beta_i = \left(\frac{\rho(x_i)}{\rho_1}\right)^{1/\nu}$. If the radius r of the balls of the first generation tends to 0, the function $\chi_{\mathcal{U}}(x) \to \rho(x)$ weakly in $L^\infty(K_0')$.

Now let us consider the domain K_2, where $|v^N| > \delta$. The whole of this domain is foliated into the integral curves of the field v^N; there they have bounded curvature, and all their derivatives are bounded too. Consider one ball, call it B, of the sand-like domain $\mathcal{U}$. Let us perform a volume-preserving change of coordinates in the neighbourhood of B (its radius is small compared to the inverse of the curvature of stream-lines), so that the stream-lines in these coordinates become parallel straight lines. If these coordinates are $(y_1, \dots, y_\nu) = (y', y_\nu)$, the equations of the stream-lines have the form $y' = \text{const}$. But the field v^N is not constant and, in the new coordinates, has the form $v = b(y)\frac{\partial}{\partial y_\nu}$. Domain B in the new coordinates is close to the ball; anyway, it is strictly convex. Let us divide B by vertical planes into rectangular bars (for simplicity we confine ourselves to the case $\nu = 3$; the case $\nu > 3$ is considered similarly). The size of these bars is small enough. Call them B_{ij}. Let us divide each bar B_{ij} into two by the surface $\Gamma_{ij} : x_1 = g(x_2, x_3)$. At the instant t_{i-1}, this surface is a plane, parallel to the (x_2, x_3)-plane, cutting a sub-bar B_{ij}^+ from the bar B_{ij}, such that the volume of $B_{ij}^+ = \rho(x_0)\operatorname{vol}(B_{ij})$ (here x_0 is the center of the ball B).

Note: Instead of the bar $B_{ij} \cap B$ with curved upper and lower surfaces, we consider the "rectangular" bar, so that the domain B becomes "steppy". The width d of the bars B_{ij} is small enough. (We are not bounded by anything in choosing the small parameters.) Let us cut the bars B_{ij} by vertical planes, parallel to $x_1 - x_3$ planes, and let us work in these planes. Let $(x_1, x_2) = (\overline{x}_1, \overline{x}_2)$ be the left side of this section. In the subdomain B_{ij}^+, consider a vector field V of the form $(A(x_3, t)(x_1 - \overline{x}_1), 0, b(\overline{x}_1, \overline{x}_2, x_3))$, where $A(x_3, t)$ is chosen in such a manner that the field V is solenoidal (we define V simultaneously and independently in all the sections of B_{ij}, for all

B_{ij} and all the balls of our sand-like domain). This means that

$$\frac{\partial b}{\partial x_3} + A(x_3) = 0 .$$

In the complementary subdomain B_{ij}^-, we define a vector field W such that entire stream through each horizontal section of B_{ij} is zero. The best way to do it is to use the stream function. If $\psi(x_1, x_3)$ is the stream function of the flow in B_{ij}^+, then $\psi = \text{const}$ (we may set if 0) on the left side of B_{ij}^+, and is linear along each horizontal section of B_{ij}^+. Let us continue it linearly with respect to x_1 in B_{ij}^-, so that it vanishes on the right side of B_{ij}^-. To define the flow at all points, we continue it piecewise-linearly in the triangles with the base on the ends of B_{ij} and height equal to the base; thus the flow is defined in the plane section of B_{ij}, with added triangles in the same plane. Now, let us do it in all the sections of B_{ij}, for all i, j, corresponding to one ball B, and for all the balls of our sand-like domain $\mathcal{U}$. Outside the union of these polygons, set the fields V and W identically zero. Let us choose the width of the bars B_{ij} so small that the added triangles do not intersect other balls of the domain $\mathcal{U}$. Now, set $R = \bigcup B_{ij}^+$, where the union i is taken over all balls of the domain $\mathcal{U}$, all bars B_{ij} and all sections of these bars, parallel to the coordinate plane (x_1, x_3) (in the local coordinate system, consistent with the field v^N).

The domain R depends on t, for it is approximately transported by the field v^N. Choosing the width of B_{ij} sufficiently small, we arrive at the flow in R that has the velocity field V, arbitrarily close to $v^N(x,t)$. The longitudinal component of V is the same as that for v^N, and the transverse component may be done arbitrarily small. R is invariant under the flow with the velocity field V. So the 1st phase (contained in R) moves arbitrarily close to the 1st phase of 2-flow $\mathcal{M}$ (on the time interval (t_{i-1}, t_i)). To prove the similar result for the domain $S_t = K \setminus (R_t \cup K_0 \cup K_1)$, observe that S_t may be decomposed into 2 domains, $S_t = S_{1,t} \cup S_{2,t}$ such that $\operatorname{mes} S_{2,t} < \delta_1$ (and δ_1 may be done arbitrarily small), $|W - w| < \delta_2$ on $S_{1,t}$ and $|w| < C$ on $S_{2,t}$, where δ_1, δ_2 may be done arbitrarily small, choosing parameters of our construction, and C does not depend on these parameters. The measure of points x in S, moving with speed W, such that they spend a total time τ from the time interval $[t_{i-1}, t_i]$ in $S_{2,t}$, is not more than $\delta_1 \cdot \frac{t_i - t_{i-1}}{\tau}$. If $\tau = \delta_1^{1/2}$, then

$$\operatorname{mes} \Sigma_2^i \leq (t_i - t_{i-1}) \cdot \delta_1^{1/2} ,$$

where

$$\Sigma_2^i = \left\{ x \in S_{t_{i-1}} \mid \operatorname{mes} \left\{ t \in [t_{i-1}, t_i] \mid Z_t(x) \in S_2 \right\} > \delta_1^{1/2} \right\} ,$$

and $Z_t : S_{t_{i-1}} \to S_t$ is the flow defined by the velocity field W for $t_{i-1} \le t \le t_i$.

On the set $Z_t(\Sigma_2^i)$ the speed W is bounded by some constant C. If $x \in \Sigma_1^i = S_{t_{i-1}} \setminus \Sigma_2^i$, then, for all $t \in [t_{i-1}, t_i]$ but the set of measure $\le \delta_1^{1/2}$,

$$\left|W\big(Z_t(x),t\big) - w^N\big(Z_t(x),t\big)\right| < \delta_2 \ ,$$

and W is bounded by C for all t. Hence, if ζ_t is the flow of the second phase of the initial 2-flow and $x \in \Sigma_1^i$, then

$$\left|\zeta_t \circ \zeta_{t_{i-1}}^{-1}(x) - Z_t(x)\right| \le C_1\delta_2(t_i - t_{i-1}) + C\delta_1^{1/2} \ .$$

Thus for all the points of S but the set of the measure $< C\delta_1^{1/2}(t_i - t_{i-1})$, the deviation of the motion Z_t (with the field W) from the motion ζ_t (with the field w^N) is less than $C(\delta_2 + \delta_1^{1/2})$, and δ_1, δ_2 may be done arbitrarily small by the choice of the parameters of construction.

Step 5.5. We constructed some discontinuous flow on the time interval $[t_{i-1}, t_i]$ $(i = 1, \dots, n)$; this flow simulates a 2-flow $\mathcal{M}$ on this interval. Now let us construct an intermediate flow connecting that on the time intervals $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$. Let τ be small enough to compare with $t_i - t_{i-1}$, and stop the previous flow at the moment $t_i - \tau$. If the flow is immobile on each interval $[t_i - \tau, t_i]$, then for $\tau \to 0$ it will tend to the flow with $\tau = 0$. So take τ small enough and let us define a *reconstruction*, i.e. a discontinuous flow ψ_t on the time interval $[t_i - \tau, t_i]$, transferring the domains $R_{t_i - \tau}, S_{t_i-\tau}$ into R_{t_i}, S_{t_i}. To do this, let us divide K into small equal cubes $K^{(j)}$ of size ℓ. Let us construct domains R_{t_i}, S_{t_i}, serving the flow for $t_i \le t < t_{i+1} - \tau$, as described above, and in such a manner that $\mathrm{mes}(R_{t_i} \cap K^{(j)}) = \mathrm{mes}(R_{t_i-\tau} \cap K^{(j)})$ (i.e. the volume of the 1st and 2nd phases in each cube $K^{(i)}$ are invariant during the reconstruction). Now let us construct a discontinuous flow in all the cubes $K^{(j)}$ (separately and independently in each), so that to transfer all of the 1st phase into the new position.

Let us divide each cube $K^{(j)}$ into M^ν equal cubes κ; call the cube κ *black*, if it is entirely contained in $R_{t_i-\tau}$. The volume occupied by the black cubes in $K^{(j)}$ tends to $\mathrm{vol}(R_{t_i-\tau} \cap K^{(j)})$ when $M \to \infty$. Let us define the black cubes for the new domain R_{t_i}. Suppose that the numbers of the black cubes in $R_{t_i-\tau} \cap K^{(j)}$ and in $R_{t_i} \cap K^{(j)}$ are equal; otherwise take the least of them and denote it by p. Take p black cubes from the domain $R_{t_i-\tau} \cap K^{(j)}$; there exists some permutation $\Psi^{(j)}$ of all the cubes κ in $K^{(j)}$, transferring these black cubes into p cubes, contained in $R_{t_i} \cap K^{(j)}$. As was proved in [S1], there exists a piecewise-smooth flow ψ_t in K, $t_{i-1}\tau \le t \le t_i$, $\psi_{t_i-\tau} = \mathrm{Id}$, $\psi_{t_i}\big|_{K^{(j)}} = \Psi^{(j)}$, ψ_t is smooth in each small cube κ, and the length of this flow is not more than $C \cdot \ell$ (where ℓ is the size of the cubes $K^{(j)}$).

Let us organize this flow simultaneously and independently in all cubes $K^{(j)}$. Then the action of this flow $J\{\psi_t\}_{t=t_i-\tau}^{t_i} \le C\,\ell^2/\tau$, and may be done arbitrarily small by the appropriate choice of ℓ. For M sufficiently large, the measure of the set of points $x \in R(t_i - \tau)$, such that their images after $\psi^{(i)}$ are contained in $R(t_i)$, may be done arbitrarily close to $\operatorname{mes} R(t_i)$. After this permutation Ψ we divide K once more into domains $R(t_i)$ and $S(t_i)$, as described above; most of $R(t_i)$ consists of the pieces of $R(t_i - \tau)$. After this the reconstruction is over, and we proceed as above. The flows Z_t for $t_{i-1} \le t \le t_i - \tau$, and the flows ψ_t, $t_i - \tau \le t \le t_i$ $(i = 1, \dots, N)$, form together a discontinuous flow $\chi_t : K \to K$, $0 \le t \le 1$, $\chi_0 = \mathrm{Id}$. K is divided into 2 domains, R_0 and S_0, trajectories $\chi_t(x)$ of most points $x \in R_0$ are close (in H^1) to the trajectories $\eta_t(x)$ of the points of the first component of the 2-flow $\mathcal{M}$; trajectories of the points of S_0 are H_1-close to the trajectories $\zeta_1(x)$ of the points of the second component of $\mathcal{M}$. The points of K, for which this is not true, form a set of small measure and the actions of trajectories of all such points are bounded. The measure of such an exclusive set, and the distance in H^1 between trajectories $\chi_t(x)$ and $\eta_t(x)$, or $\zeta_t(x)$, may be done arbitrarily small by choosing the parameters of construction.

Step 5.6. Now we shall construct a smooth flow, approximating a discontinuous flow χ_t. Let $z(x,t) = \dot{\chi} \circ \chi_t^{-1}(x)$ be an Eulerian velocity field of discontinuous flow χ_t. Let us choose a function $\varphi(x) = \varphi(|x|) \in C_0^\infty$, such that $\varphi(x) \ge 0$, $\varphi(x) = 0$, if $|x| > 1$ and $\int \varphi(x)dx = 1$. Let $z_\varepsilon(x,t)$ be a smoothed velocity field:

$$z_\varepsilon(x,t) = \int z(y,t)\varphi_\varepsilon(x-y)dy \ ,$$

where $\varphi_\varepsilon(x) = \varepsilon^{-\nu}\varphi(\frac{x}{\varepsilon})$.

Let χ_t^ε be a flow in K generated by the vector field $z_\varepsilon(x,t)$. If $\varepsilon \to 0$, then $z_\varepsilon \to z$ in C^∞ in each domain where z is smooth. The field z transports its smoothness; i.e. for almost all $x \in K$, $z(x,t)$ is smooth in the neighbourhood of its trajectory $(\chi_t(x), t)$ $(0 \le t \le 1)$. Therefore, for each $\delta > 0$,

$$\lim_{\varepsilon\to 0} \operatorname{mes}\left\{x \in K \;\middle|\; \left|\chi_t(x) - \chi_t^\varepsilon(x)\right| < \delta, \left|\dot{\chi}_t(x) - \dot{\chi}_t^\varepsilon(x)\right| < \delta \ , \ 0 \le t \le 1\right\} = 1 \ ,$$

and $|\dot{\chi}_t^\varepsilon(x)| < C$, $|\dot{\chi}_t(x)| < C$ for all x. Hence, χ_t^ε also approximates the 2-flow $\mathcal{M}$ together with J.

Step 6. Now we shall approximate the generic multiflow $\mathcal{M} = (a_1, \dots, a_p; \xi_{1,t}, \dots, \xi_{p,t})$ by a smooth flow, using representation of $\mathcal{M}$ as a composition of 2-flows.

We shall construct this approximation inductively. Assume that for each p-flow $\mathcal{M} = (a_1, \dots, a_p; \xi_{1,t}, \dots, \xi_{p,t})$ we can construct a smooth flow $\psi_t \subset \mathcal{D}$, approximating it in weak $*$ topology together with J.

Let $\mathcal{M}' = \mathcal{M} * \mathcal{M}^p$ where $\mathcal{M}^p = (a^p, b^p; \eta_t^p, \zeta_t^p)$ is a smooth 2-flow. Let χ_t^p be a smooth flow, approximating the 2-flow $\mathcal{M}^p$, constructed in Step 5. Then the flow $\mathcal{M}'$ may be approximated by the p-flows of the form

$$\mathcal{M}'' = (a_1, \dots, a_p; \xi_{1,t}, \dots, \xi_{p-1,t}, \xi_{p,t} \circ \chi_t^p) .$$

This is a smooth p-flow, which, by the inductive assumption, may be approximated by the smooth flow ψ_t. But each $(p+1)$ flow may be represented as a composition of a p-flow and a 2-flow, as was proved above. So, each multiflow may be approximated by (a sequence of) the smooth flows in K.

Step 7. We started our construction from the approximation of a given GF μ, connecting Id and ξ, by a GF μ^i connecting Id and ξ, such that μ^i-almost all trajectories $x(t)$ are t-independent for $1 - i^{-1} \leq t \leq 1$.

In Steps 2-6 we have constructed a sequence of smooth flows $\phi_t^{i,j}$, approximating GF μ^i together with J and t-independent for $q - i^{-1} \leq t \leq 1$. But these flows do not, in general, satisfy the boundary condition: $\psi_{1-i^{-1}}^{i,j} \not\equiv \xi$.

LEMMA 1.3. *If $\psi_t^{i,j}$ is a sequence of smooth incompressible flows in K $(j = 1, 2, \dots)$, such that $\mu_{\psi_t^{i,j}} \underset{j\to\infty}{\longrightarrow} \mu^i$ weak $*$, then $\psi_{1-i^{-1}}^{i,j} \underset{j\to\infty}{\longrightarrow} \xi$ in $L^2(K, \mathbf{R}^\nu)$.*

Proof: Consider the function

$$f\{x(t)\} = \left|\xi(x(0)) - x(t - i^{-1})\right|^2 .$$

For the GF μ^i, $\int f\{x\}\mu^i\{dx\} = 0$, because GF μ^i satisfies the boundary condition. By the definition of weak $*$-convergence,

$$\int_\Omega f\{x\}\mu_{\psi_t^{i,j}}\{dx\} = \int_K \left|\xi(x) - \psi_{1-i^{-1}}^{i,j}(x)\right|^2 dx = \|\psi_{1-i^{-1}}^{i,j} - \xi\|_{L^2}^2 \underset{j\to\infty}{\longrightarrow} 0 . \ \square$$

If $\|\psi_{1-i^{-1}}^{ij} - \xi\|_{L^2} = \delta_j$ and $\nu \geq 3$, it was proved in [S1] that

$$\mathrm{dist}_{\mathcal{D}(K^\nu)}(\psi_{t-i^{-1}}^{i,j}, \xi) \leq C\delta_j^\alpha ,$$

where $C > 0$, $\alpha > 0$ depend only on ν. Therefore, there exists a path $\psi_t^{i,j}$, $1 - i^{-1} \leq t \leq 1$, such that $\psi_1^{i,j} = \xi$, and $J\{\psi_t^{i,j}\}_{1-i^{-1}}^1 \leq \frac{1}{2}C^2\delta_j^{2\alpha} \cdot i$; thus, if $\delta_j \xrightarrow[j\to\infty]{} 0$, the sequence of the smooth flows $\psi_t^{i,j}$ satisfies the boundary conditions $\psi_0^{i,j} = \mathrm{Id}$, $\psi_1^{i,j} = \xi$, $\mu_{\psi_t^{i,j}} \xrightarrow{w^*} \mu^i$, and

$$J\{\psi_t^{i,j}\}_0^1 \to J\{\mu^i\}_0^1 \qquad (j \to \infty) .$$

GF μ^i, in its turn, approximates GF μ together with the action. Thus the Approximation Theorem is proved.

2. Applications of the Approximation Theorem

2.1 Estimate of Distances in $\mathcal{D}$. The Approximation Theorem makes evident the following assertion.

LEMMA 2.1. *If $\nu \geq 3$, then for each $\xi \in \mathcal{D}$* $\mathrm{dist}(\mathrm{Id}, \xi) = \inf(2 \cdot J\{\mu\}_0^1)^{1/2}$, *where* inf *is taken over all* GF μ, *connecting* Id *and* ξ.

In fact, for each GF μ, connecting Id and ξ, there exists a sequence of smooth flows ξ_t^i, connecting Id and ξ and such that $\mu_{\xi^i} \xrightarrow{w^*} \mu$, $J\{\xi_t^i\}_0^1 \to J\{\mu\}_0^1$. But $L\{\xi_t^i\}_0^1 \leq (2J\{\xi_t\}_0^1)^{1/2}$, and the equality may be attained after some change of variable t.

Our first result, concerning the geometry of $\mathcal{D}(K^\nu) = \mathcal{D}$, is the following.

THEOREM 2.1. *If $\nu \geq 3$, and K^ν is a unit ν-dim cube, then*

$$\mathrm{diam}\, \mathcal{D} \leq 2\sqrt{\frac{\nu}{3}} .$$

Proof: Our construction is almost identical to that of Brenier [Br1], who proved that all fluid configurations on the torus are attainable by GF. For each $\xi \in \mathcal{D}$, we construct a GF μ, connecting Id and ξ, such that $J\{\mu\}_0^1 = \frac{2\nu}{3}$; then our main theorem states that for each $\varepsilon > 0$ there exists a smooth flow σ_t, connecting Id and ξ, and such that $J\{\sigma_t\}_0^1 < \frac{2\nu}{3} + \varepsilon$. After some reparametrization of σ_t, $L\{\sigma_t\} \leq \sqrt{\frac{4\nu}{3} + 2\varepsilon}$; this just means that $\mathrm{diam}\, \mathcal{D} \leq 2\sqrt{\frac{\nu}{3}}$. Suppose that the cube K is defined as follows:

$$K = \left\{x \in \mathbf{R}^\nu \;\middle|\; |x_i| < \tfrac{1}{2} \;,\; i = 1, \ldots, \nu\right\} .$$

Let Γ be a discrete group of motions of $\mathbf{R}^\nu$, generated by the reflections in the faces of K. For each $y \in K$, $v \in K$, we define a path $x_{y,v}(t) \subset K$, $0 \leq t \leq \frac{1}{2}$:

$$x_{y,v}(t) = \Gamma(y + 4tv) \cap K .$$

This is a billiard trajectory in K. The mapping $\phi_y : v \longrightarrow x_{y,v}\left(\frac{1}{2}\right)$ is a 2^ν-fold covering of K, and ϕ_y is volume-preserving. Let K_y, K_v, K_z, K_w be 4 copies of K with coordinates y, v, z, w correspondingly (i.e. $K_y = \left\{y \in \mathbf{R}^\nu_y \mid |y_i| < \frac{1}{2}\right\}$, etc.) Let us define a set $\Omega \subset K_y \times K_v \times K_z \times K_w$; $\omega = (y, v, z, w) \in \Omega$, if $z = \xi(y)$, and $x_{y,v}\left(\frac{1}{2}\right) = x_{z,w}\left(\frac{1}{2}\right)$.

For each $y \in K$, let $S_y = (\{y\} \times K_v \times \{\xi(y)\} \times K_w) \cap \Omega$. This is a graph of a correspondence in $\{y\} \times K_v \times \{\xi(y)\} \times K_w$, that is a 2^ν-fold covering of both $\{y\} \times K_v$ and $\{\xi(y)\} \times K_w$. This correspondence is locally a motion, and $\Omega = \bigcup_{y \in K_y} S_y$. Let $d\omega = 2^{-\nu} dy\, dv$ be a normed volume element on Ω. Then the required GF μ is a random process with probability space $(\Omega, d\omega)$ and is defined as

$$x(t,\omega) = \begin{cases} x_{y,v}(4t) \ , & 0 \le t \le \frac{1}{2} \\ x_{\xi(y),w}(4 - 4t) \ , & \frac{1}{2} \le t \le 1 \ , \end{cases}$$

where $\omega = (y, v, \xi(y), w) \in \Omega$. This GF may be described as follows: each fluid particle is split at $t = 0$ into a continuum of parts, all the parts of a particle, situated at $t = 0$ at the point $y \in K_1$, move independently along the billiard trajectories in K and at $t = \frac{1}{2}$ they fill K uniformly. After $t = \frac{1}{2}$ they start moving along other billiard trajectories and at $t = 1$ they all arrive at the point $\xi(y)$. All the particles split and move independently in the same manner: homogeneity is fulfilled automatically.

The action of this GF

$$J\{\mu\} = \frac{1}{2}\int_{K_v} 16v^2 dv = \nu \cdot \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} 16x^2 dx = \frac{2\nu}{3}$$

$$L\{\mu\} = \sqrt{2J} = 2\sqrt{\frac{\nu}{3}}$$

Thus, the estimate of the diam $\mathcal{D}$ is proved.

2.2 Estimate of the distances between the elements of $\mathcal{D}$. $\mathcal{D}$ is isometrically embedded into the Hilbert space $L^2(K^\nu, \mathbf{R}^\nu)$, and there is a "coarse" distance between its elements:

$$\delta(\xi,\eta) = \left\|\xi(x) - \eta(x)\right\|_{L^2}$$

Is there any connection between the distances dist and δ? In [S1] it was proved that if $\nu \ge 3$, there exist $C = C(\nu) > 0$, $\alpha = \alpha(\nu) > 0$, such that

$$\mathrm{dist}(\xi,\eta) \le C\delta^\alpha(\xi,\eta) \ .$$

Direct construction of [S1] provides a very weak estimate for α; for example, if $\nu = 3$, then $\alpha \ge \frac{1}{64}$. Our approximation theorem enables us to improve this estimate.

THEOREM 2.2. *If $\nu \geq 3$, then for all $\xi, \eta \in \mathcal{D}$,*

$$\operatorname{dist}(\xi, \eta) \leq C\delta^{\alpha}(\xi, \eta) ,$$

where $\alpha \geq \frac{2}{\nu+4}$.

Proof: It is sufficient to prove this estimate for $\eta = \mathrm{Id}$, because both dist and δ are right invariant metrics.

Let us denote by $\Delta(x)$ the function $|\xi(x)-x|$, and suppose that $\|\Delta(x)\|^2_{L^2} = \delta^2$ is sufficiently small. We shall construct a GF μ, connecting Id and ξ, and such that $J\{\mu\}_0^1 \leq C\delta^{\frac{4}{\nu+4}}$. Instead of the cube K, let us consider the ν-dimensional torus $\mathbf{T}^\nu$. In fact, this is equivalent. Really, let us consider the group Γ of motions in $\mathbf{R}^\nu$, generated by reflections in the faces of the unit cube $K \subset \mathbf{R}^\nu$ and if the subgroup Γ' consisted of the parallel shifts. The fundamental domain of Γ' is a cube $2K$, that may be subdivided into 2^ν shifts of K. Each $\xi \in \mathcal{D}(K)$ may be prolonged to the mapping $\widetilde{\xi}$ of $2K$ onto itself by conjugation: $\widetilde{\xi}(\gamma x) = \gamma \circ \xi \circ \gamma^{-1}(x)$, $\gamma \in \Gamma$. It is evident the $\xi(x+b) = \xi(x)+b$ for each vector $b \in 2\mathbf{Z}$ and, therefore, ξ may be regarded as a mapping of the torus $R^\nu/2\mathbf{Z}$ into itself.

For each trajectory $\widetilde{x}(t)$ on $\mathbf{T}$ we may construct a trajectory $x(t) = \Gamma x(t) \cap K = \pi(x(t))$. Let $\widetilde{\mu}$ be a GF on $\mathbf{T}$, connecting Id and $\widetilde{\xi}$; then $\mu = \pi(\widetilde{\mu})$, the image of the measure μ under the transformation π, is a GF in K, connecting Id and ξ. Evidently, $J(\mu) = J(\widetilde{\mu})$.

So, let $\widetilde{\xi} \in \mathcal{D}(\mathbf{T})$; we shall construct a GF $\widetilde{\mu}$, connecting Id and $\widetilde{\xi}$, and such that $L\{\mu\} \leq C\delta^{\frac{2}{\nu+4}}$, where $\delta = \|\Delta\|_{L^2} = \|\xi(x) - x\|_{L^2}$.

Let us take as a probability space Ω the tangent space $T\mathbf{T}^\nu$, (x,p) are the coordinates in Ω. Measure $\widetilde{\mu}$ in Ω has a density $2^{-\nu}\varepsilon^{-\nu}\varphi(\frac{p}{\varepsilon})$, where $\varphi(p) \in C_0^\infty$, $\varphi(p) = \varphi(|p|)$, $\varphi(p) \geq 0$, $\varphi = 0$, if $|p| \geq 1$; $\varphi(p) \equiv 1$, if $0 < p < \frac{1}{2}$, and $\int \varphi(p)dp = 1$. For each $\omega = (x,p) \in \Omega$, let us define a trajectory on $\mathbf{T}^\nu$,

$$x(t,\omega) = \begin{cases} x + 4tp , & 0 \leq t \leq \frac{1}{4} \\ x + p + 2(t - \frac{1}{4})(\widetilde{\xi}(x) - x) , & \frac{1}{4} \leq t \leq \frac{3}{4} \\ \xi(x) + 4(1-t)p , & \frac{3}{4} \leq t \leq 1 . \end{cases} \tag{2.1}$$

This GF may be described as follows: at the moment $t = 0$ each particle x splits into continuum of particles (x,p); they spread into a blob of size ε; all these blobs are moving with constant speed to their final positions with the center in $\widetilde{\xi}(x)$, and then they concentrate into a single particle at the point $\widetilde{\xi}(x)$.

This flow is not in fact satisfactory, for it is not incompressible. So, we must introduce a correcting mapping $f_t : \mathbf{T} \to \mathbf{T}$, $\frac{1}{4} \le t \le \frac{3}{4}$ (on the 1st and the 3rd stages our GF (2.1) is incompressible). In the case of torus construction of the mapping f_t is much simpler and more explicit than in the general case (see [Mo1]). Let $x_1, \dots, x_\nu$ be the coordinates on $\mathbf{T}$ $(0 \le x_i < 2)$. Let $\rho(x)$ be arbitrary a smooth positive normed density. Then we may construct transformation $f : \mathbf{T} \to \mathbf{T}$, translating the measure $\rho\, dx$ into the measure with constant density, and having "triangular"form:

$$
\begin{aligned}
&f(x_1, \dots, x_\nu) = (y_1, \dots, y_\nu) \\
&y_1 = f_1(x_1) \;; \\
&y_2 = f_2(x_1, x_2) \;; \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&y_\nu = f_\nu(x_1, \dots, x_\nu) \;.
\end{aligned}
$$

The functions $f_1, \dots, f_\nu$ may be found from the following relations, expressing the fact that $f^{*-1}(\rho\, dx) = dx$,

$$
\begin{aligned}
&\frac{\partial f_1}{\partial x_1} = \int \rho(x_1, \dots, x_\nu) dx_2 \dots dx_\nu \;, \\
&\frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} = \int \rho(x_1, x_2, \dots, x_\nu) dx_3 \dots dx_\nu \;, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_\nu}{\partial x_\nu} = \rho(x_1, \dots, x_\nu) \;;
\end{aligned}
$$

The integration constants may be chosen arbitrarily; we choose them so that the centers of mass of the $(\nu-1)$- dim, $(\nu-2)$- dim, ... sections remain unchanged.

Let us estimate $\frac{\partial f}{\partial t}$. If $\rho = \rho(x,t)$, then

$$
\begin{aligned}
&\frac{\partial}{\partial x_1} \frac{\partial f_1}{\partial t} = \frac{\partial}{\partial t} \int \rho(x_1, \dots, x_\nu, t) dx_2 \dots dx_\nu \;; \\
&\frac{\partial}{\partial x_2} \frac{\partial f_2}{\partial t} = \frac{\partial}{\partial t} \left[\frac{\partial f_1}{\partial x_1}\right]^{-1} \int \rho(x_1, \dots, x_\nu, t) dx_3 \dots dx_\nu \;; \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\frac{\partial}{\partial x_i} \frac{\partial f_i}{\partial t} = \frac{\partial}{\partial t} \left[\frac{\partial f_1}{\partial x_1}\right]^{-1} \cdots \left[\frac{\partial f_{i-1}}{\partial x_{i-1}}\right]^{-1} \int \rho(x_1, \dots, x_\nu, t) dx_{i+1} \dots dx_\nu \;; \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\frac{\partial}{\partial t} \frac{\partial f_\nu}{\partial x_\nu} = \left[\frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_{\nu-1}}{\partial x_{\nu-1}}\right]^{-1} \rho(x_1, \dots, x_\nu, t) \;.
\end{aligned}
$$

It is easy to observe that if $0 < c_1 < \rho(x,t) < c_2$, the operators $F_t : \frac{\partial \rho}{\partial t} \to \frac{\partial f}{\partial t}$ are uniformly bounded in L^2 (they have uniformly bounded kernel) and, therefore,

$$J\{\widetilde{\mu}\}_{1/4}^{3/4} \leq C \int_{1/4}^{3/4} \left\| \frac{\partial \rho}{\partial t} \right\|_{L^2}^2 dt \ .$$

But the bounds for $\frac{\partial \rho}{\partial t}$ and ρ depend on the parameter ε; we have to choose it in such a manner that ρ us uniformly bounded from both sides. Let us estimate it. If $\frac{1}{4} \leq t \leq \frac{3}{4}$ and the GF $\widetilde{\mu}$ is defined by (2.1), $\rho(x,t)$ may be estimated as follows:

$$C\varepsilon^{-\nu} \operatorname{mes} \left\{ x \;\middle|\; |x+\tau\Delta(x)-y| < \frac{\varepsilon}{2} \right\} \leq \rho(y,t) \\ \leq C\varepsilon^{-\nu} \operatorname{mes} \left\{ x \;\middle|\; |x+\tau\Delta(x)-y| < \varepsilon \right\} ,$$

where $\Delta(x) = \xi(x) - x$, $\tau = 2(t - \frac{1}{4})$.

Now,

$$\rho(y,t) \leq C\varepsilon^{-\nu} \operatorname{mes} \left\{ x \;\middle|\; |x-y| \leq C_1\varepsilon \right\} + \\ + C\varepsilon^{-\nu} \operatorname{mes} \left\{ x \;\middle|\; |x-y| > C_1\varepsilon, \text{ and } |\Delta(x)| > C_1\varepsilon \right\} .$$

If $\delta^2 = \int |\Delta(x)|^2 dx$, then

$$\rho(y,t) \leq C\varepsilon^{-\nu} \left(\varepsilon^\nu + \frac{\delta^2}{\varepsilon^2} \right) ,$$

and if $\varepsilon \geq c\delta^{\frac{2}{\nu+2}}$, then

$$\rho(y,t) < \text{const} \ .$$

In a similar manner,

$$\rho(y,t) \geq c\varepsilon^{-\nu} \left[\operatorname{mes} \left\{ x \;\middle|\; |x-y| < \frac{\varepsilon}{2} \right\} \right. \\ \left. - \operatorname{mes} \left\{ x \;\middle|\; |x-y| < \frac{\varepsilon}{4} \ , \ |\Delta(x)| > \frac{\varepsilon}{4} \right\} \right] \\ \geq C\varepsilon^{-\nu} \left(\varepsilon^\nu - C\frac{\delta^2}{\varepsilon^2} \right) ;$$

if we set $\varepsilon > C\delta^{\frac{2}{\nu+2}}$, then

$$\rho(y,t) > \text{const} > 0 \ .$$

Now, we may estimate $\frac{\partial \rho}{\partial t}$. $\rho(y,t) = \int \rho_x(y,t)dx$, where

$$\rho_x(y,t) = c\varepsilon^{-\nu}\varphi\left(\frac{y-x-\tau\Delta(x)}{\varepsilon}\right) \; ;$$

$$\left\|\frac{\partial \rho_x}{\partial t}\right\|^2_{L^2} \le c\varepsilon^{\nu}\cdot\varepsilon^{-2\nu-2}\cdot\left|\Delta(x)\right|^2 \; ;$$

$$\begin{aligned}\left\|\frac{\partial \rho}{\partial t}\right\|_{L^2} &\le \int\left\|\frac{\partial \rho_x}{\partial t}\right\|_{L^2} dx \le \\ &\le c\varepsilon^{-\frac{\nu+2}{2}}\int\left|\Delta(x)\right|dx \le \\ &\le c\varepsilon^{-\frac{(\nu+2)}{2}}\|\Delta\|_{L^2} = C\varepsilon^{-\frac{(\nu+2)}{2}}\cdot\delta \; ;\end{aligned}$$

if $\varepsilon > C\delta^{\frac{2}{\nu+2}}$, then $0 < C_1 < \rho(y) < C_2$, and

$$\left\|\frac{\partial}{\partial t}f_t\right\|_{L^2} \le C\varepsilon^{\frac{-(\nu+2)}{2}}\cdot\delta \; .$$

Hence

$$\begin{aligned}L\{\mu\}_0^1 &= L\{\mu\}_0^{1/4} + L\{\mu\}_{1/4}^{3/4} + L\{\mu\}_{3/4}^1 \le \\ &\le C\varepsilon + C\varepsilon^{-\frac{(\nu+2)}{2}}\cdot\delta \; .\end{aligned}$$

In order to optimize the estimate for $L\{\mu\}$, we take $\varepsilon = C\varepsilon^{-\frac{(\nu+2)}{2}}\cdot\delta$; this gives $\varepsilon = C\delta^{\frac{2}{\nu+4}} > \delta^{\frac{2}{\nu+2}}$ for small δ. This means that, with our choice of ε, ρ remains bounded from both sides and our estimates are self-consistent. Thus Theorem 2.2 is proved.

2.3 The new proof of the fact that if $\nu = 2$, then $\operatorname{diam}\mathcal{D}(K) = \infty$.

Our result is a little more precise.

THEOREM 2.3. *Let $G(t) \subset R^2$ be a domain of area S, depending smoothly on t, $0 \le t \le 1$; let $\xi_t : G(0) \to G(t)$ be a family of the area preserving diffeomorphisms, such that for each two points $x_1, x_2 \in G(0)$, their images $\xi_t(x)$ and $\xi_t(x_2)$ make at least N revolutions around each other, when t passes from 0 to 1. Then*

$$J\{\xi_t\}_0^1 \ge CS^2\cdot N \; ,$$

where

$$J\{\xi_t\}_0^1 = \int_0^1 dt\int_{G(0)}\frac{1}{2}\left|\frac{\partial \xi_t(x)}{\partial t}\right|^2 dx$$

is the action of the flow ξ_t.

Proof: 1. Let us do a homotopy with coefficient $S^{-\frac{1}{2}}$. It will reduce the problem to the case $S = 1$, and remove the factor S^2 from the estimate of J.

2. Let us reduce the problem to a more simple one for the generalized flows. Consider the moving domain $\widetilde{G}_t = G_t \times G_t \subset \mathbf{R}^2 \times \mathbf{R}^2$; let Δ be a diagonal in $\mathbf{R}^2 \times \mathbf{R}^2$. For each 2 points $x_1, x_2 \in G_0$, denote $\widetilde{x} = (x_1, x_2)$ and consider the trajectory

$$\widetilde{\xi}_t(\widetilde{x}) = \big(\xi_t(x_1), \xi_t(x_2)\big) \subset \mathbf{R}^2 \times \mathbf{R}^2 .$$

It is clear, that

$$J\{\widetilde{\xi}_t\}_0^1 = 2J\{\xi_t\}_0^1 .$$

The flow $\widetilde{\xi}_t$ in $\widetilde{G}_t$ is evidently incompressible, the diagonal Δ is invariant and each point $\widetilde{\xi}_t(\widetilde{x})$ makes at least N revolutions around Δ $(0 \le t \le 1)$. The volume of $\widetilde{G}_t$ is equal to 1, and the area of the intersection of $\widetilde{G}_t$ and arbitrary plane Δ_h, parallel to Δ, does not exceed 2 (in fact this plane consists of the points having the type $(x, x+h)$; the area of $\widetilde{G}_t \cap \Delta_h$ is equal to the doubled area of $G_t \cap (G_t + h)$). Thus the measure of δ-neighbourhood of Δ, intersected by $\widetilde{G}_t$, does not exceed $2\pi\delta^2$. Let us estimate $J\{\widetilde{\xi}_t\}_0^1$. Let E be orthogonal complement to Δ in $\mathbf{R}^4$. If we take a projection of each trajectory $\widetilde{\xi}_t(x)$ on E, then we get a GF $\mu\{dy\}$ in E (with the probability space $(\Omega = \widetilde{G}(0))$ such that

1. for each domain $A \subset E$, and each $t_0 \in [0,1]$,

$$\mu\{y(t_0) \in A\} \le 2\,\mathrm{mes}(A) ;$$

2. $\mu\{X\} = 1$;
3. μ-almost all trajectories make at least N revolutions around the origin in E.

We are going to prove

THEOREM 2.4. *For each* GF μ, *satisfying* (1)–(3),

$$J\{\mu\}_0^1 \ge C \cdot N ,$$

where

$$J\{\mu\} = \int_\Omega \mu(dy) \int_0^1 \tfrac{1}{2}\big|\dot{y}(t)\big|^2 dt .$$

3. We start the proof of Theorem 2.4 with a sort of symmetrization. Let μ_φ be a measure obtained from μ by rotation through the angle φ around

the origin. Let $\overline{\mu}$ be a measure obtained by averaging μ_φ over all $\varphi \in [0, 2\pi)$. Then $J\{\overline{\mu}\}_0^1 = J\{\mu\}_0^1$, and the measure $\overline{\mu}$ is axisymmetric.

4. Let us represent each trajectory $y(t)$ in polar coordinates: $y(t) = (r(t), \varphi(t))$. Measure $\overline{\mu}$ is invariant under the transformation $(r, \varphi) \to (r, \varphi + \varphi_0)$ for each fixed φ_0. Let $(r(t), \varphi(t))$ be some trajectory $0 \leq t \leq 1$. We shall construct another trajectory $(r(t), \psi(t))$, such that

1. $\int_0^1 d\psi(t) = \int_0^1 d\varphi(t)$;
2. $J\{(r(t), \psi(t))\}_0^1$ is minimal.

If we replace the trajectories $(r(t), \varphi(t))$ by $(r(t), \psi(t))$, the density of particles will not change.

This is a variational problem: find a function $\psi(t)$, such that $J = \frac{1}{2}\int_0^1 [\dot{r}^2(t) + r^2(t)\dot{\psi}^2(t)]dt$ is minimal provided $\int_0^1 d\psi(t) = \int_0^1 d\varphi(t) \geq N$. The Euler Equation is

$$r^2(t)\dot{\psi}(t) = C\ ;$$

$$\dot{\psi} = \frac{C}{r^2(t)}\ ,$$

and

$$C\int_0^1 \frac{dt}{r^2(t)} = \int_0^1 d\varphi(t) \geq N\ ;$$

this implies

$$C \geq N \Big/ \int_0^1 \frac{dt}{r^2(t)}\ ,$$

and the estimate for the action:

$$\begin{aligned} J\{y(\cdot)\}_0^1 &\geq \frac{1}{2}\int_0^1 \dot{r}^2(t)dt + \frac{1}{2}\int_0^1 \frac{C^2dt}{r^2(t)} \geq \\ &\geq \frac{1}{2}\int_0^1 \dot{r}^2(t)dt + \frac{1}{2}\frac{N^2}{\int_0^1 \frac{dt}{r^2(t)}}\ , \end{aligned}$$

$\overline{\mu}$ is an axisymmetric measure on the functions $y(t) = (r(t), \varphi(t))$; if we neglect the second variable $\varphi(t)$, we obtain a measure ν on the space of scalar functions $r(t)$ (ν is projection of $\overline{\mu}$ on the first component).

Denote the modified action of $r(t)$ by $J_N\{r(\cdot)\}_0^1$:

$$J_N\{r(\cdot)\}_2^1 = \frac{1}{2}\int_0^1 \dot{r}^2(t)dt + \frac{N^2}{2\int_0^1 \frac{dt}{r^2(t)}}\ .$$

Then the action of GF $\overline{\mu}$ is estimated from below by the modified action of GF ν:

$$J\{\overline{\mu}\}_0^1 \geq J_N\{\nu\}_0^1 = \int_\Omega J_N\{r(\cdot)\}_0^1 \nu\{dr\} .$$

Measure ν satisfies the "one sided incompressibility" condition

$$\nu\{r(t_0) \in A\} \leq 4\pi \int_A r\,dr ,$$

where $A \subset \mathbf{R}^+$.

Let us estimate $J_N\{\nu\}_0^1$ from below. Suppose that $J_N\{\nu\}_0^1 < B$; then

$$\nu\{r(\cdot) \mid J_N\{r\}_0^1 < 2B\} > \tfrac{1}{2} .$$

Let $\chi(r(\cdot))$ be a characteristic functional of the set

$$\mathcal{M} = \{r(\cdot) \mid J_N\{r\}_0^1 < 2B\} ,$$

and the measure $\kappa = \chi(r) \cdot \nu$; then $\kappa(\mathcal{M}) \geq \frac{1}{2}\nu(\Omega) = \frac{1}{2}$, and

$$\kappa\{r(\cdot) \mid r(t_0) \in A\} \leq 4\pi \int_A r\,dr . \tag{2.2}$$

For κ-almost all trajectories $r(t)$,

$$\int_0^1 \dot{r}^2(t)dt \leq 2B ; \tag{2.3}$$

$$\frac{N^2}{\int_0^1 \frac{dt}{r^2(t)}} \leq 2B ,$$

or

$$\int_0^1 \frac{dt}{r^2(t)} \geq \frac{N^2}{2B} . \tag{2.4}$$

Condition (2.2) means that

$$\alpha = \kappa\{r(\cdot) \mid r(0) < a\} \leq 2\pi a^2 ;$$

if $a < a_0 = (8\pi)^{-1/2}$, then $\alpha < \frac{1}{4}$, and

$$\kappa\{r(\cdot) \mid r(0) > a_0\} > \tfrac{1}{4} .$$

For a given $r_0 > 0$ consider the (measurable) set

$$\mathcal{M}(r_0) = \left\{ r(\cdot) \mid r(0) > a_0 \ , \ \min_{0 \le t \le 1} r(t) \le r_0 \right\} .$$

Now observe that

$$\int_{\mathcal{M} \setminus \mathcal{M}(r_0)} \kappa\{dr\} \int_0^1 \frac{dt}{r^2(t)} \le \int_{r_0}^R \frac{4\pi}{r} dr = \\ = 4\pi \ln \frac{R}{r_0} \ ,$$

where R is some upper bound for $r(t)$ $(0 \le 1)$. And (2.4) implies that

$$4\pi \ln \frac{R}{r_0} \ge \kappa\big(\mathcal{M} \setminus \mathcal{M}(r_0)\big) \cdot \frac{N^2}{2B} \ ;$$

$$\kappa\big(\mathcal{M} \setminus \mathcal{M}(r_0)\big) \le 8\pi \ln \frac{R}{r_0} \cdot \frac{B}{N^2} \ .$$

If $|\ln r_0| \le C_1 \frac{N^2}{B}$, then $\kappa(\mathcal{M} \setminus \mathcal{M}(r_0)) < \frac{1}{4}$, and $\kappa \mathcal{M}(r_0) \ge \frac{1}{4}$.

This means that a large share the trajectories $r(t) \in \mathcal{M}$ (say, more than a quarter), starting from the point $r(0) > a_0$, reaches the point r_0 for some $t \in [0,1]$.

5. For each $r > 0$, $t \in [0,1]$, let $\chi_{r,t}$ be a characteristic functional of the set $\{r(\cdot) \mid r(t) < r\}$; then we may define a non-negative measure

$$\kappa_{r,t} = \frac{\partial}{\partial r} [\chi_{r,t} \cdot \kappa]$$

in $\mathcal{M}$. Condition (2.2) implies, that $\kappa_{r,t} \mathcal{M} < C \cdot r$. The measure $\kappa_{r,t}$ is concentrated on the trajectories $r(t)$, passing the point r at moment t. The radial part of the action of the GF κ is

$$J_R\{\kappa\} = \int_{\mathcal{M}} \kappa(dr) \int_0^1 \frac{\dot r(t)}{2} dt \ge \\ \ge \int_0^1 dt \int_{r_0}^{a_0} dr \int_{\mathcal{M}(r_0)} \tfrac{1}{2} \dot r^2(t) \kappa_{r,t}(dr) \ .$$

In order to estimate $J_R(\kappa)$ from below, let us introduce a modified GF. If $r(t) \in \mathcal{M}(r_0)$, then set

$$\Pi r(t) = \begin{cases} r(t) \ , & \text{if } r(\tau) < r_0 \text{ for all } \tau \le t \\ r_0 \ , & \text{if } r(\tau) = r_0 \text{ for some } \tau \le t \ . \end{cases}$$

Let us define a new measure

$$\widetilde{\kappa}_{r,t} = \kappa_{r,t} \cdot \chi_{\{\Pi r(t) = r(t)\}} \ ;$$

this means that we take into account only the trajectories that are passing the point r at time t, but have not reached r_0 before this.

It is clear that

$$J_R\{\kappa\} \geq \int_0^1 dt \int_{r_0}^{a_0} dr \int_{\mathcal{M}(r_0)} \tfrac{1}{2}\dot{r}^2(t)\widetilde{\kappa}_{r,t}\{dr\} \ .$$

Let $\widetilde{q}(r,t) = \int_\Omega \dot{r}(t)\widetilde{\kappa}_{r,t}\{dr\}$. This is a flow rate through the point r of the particles that have not reached the point r_0 before t. From the definition of $\mathcal{M}(r_0)$ we see, that

$$\int_0^1 \widetilde{q}(r,t)dt \geq C$$

for all $r \in (r_0, a_0)$. (All the particles from $\mathcal{M}(r_0)$ pass the point r at least once.)

(2.2) implies that $\int \widetilde{\kappa}\{dr\} < C \cdot r$; by the Schwartz inequality

$$\Big(\int_0^1 \int_{\mathcal{M}(r_0)} \dot{r}(t)\widetilde{\kappa}_{r,t}\{dr\}dt\Big)^2 \leq \Big(\int_0^1 \int_{\mathcal{M}(r_0)} \dot{r}^2(t)\widetilde{\kappa}_{r,t}\{dr\}dt\Big)^{1/2} \cdot \big(\widetilde{\kappa}(\mathcal{M}(r_0))\big)^{1/2} ,$$

and this implies that

$$\int_0^1 \int_{\mathcal{M}(r_0)} \tfrac{1}{2}\dot{r}^2\widetilde{\kappa}_{r,t}\{dr\}dt \geq Cr^{-1} \ .$$

Consequently,

$$\begin{aligned} J_R\{\kappa\} &= \int_{\mathcal{M}} \int_0^1 \tfrac{1}{2}\dot{r}^2\kappa\{dr\}dt \geq \\ &\geq \int_{\mathcal{M}(r_0)} \int_{r_0}^{a_0} \int_0^1 \tfrac{1}{2}\dot{r}^2(t)\widetilde{\kappa}_{r,t}\{dr\}dr\, dt \geq C|\ln r_0| \ . \end{aligned}$$

But $\ln r_0$ was defined above as $C\frac{N^2}{B}$ for some $C > 0$. Hence $J\{\mu\}_0^1 \geq J_R\{\kappa\}_0^1 \geq C\frac{N^2}{B}$. But we started with conjecture that $J\{\mu\} < B$. These inequalities do not contradict each other only if $B \geq C \cdot N$. Thus, Theorem 2.4 is proved; it implies Theorem 2.3.

COROLLARY. *If* $\nu = 2$, *then* $\operatorname{diam} \mathcal{D} = \infty$.

Proof: Consider a circle B or radius $\rho > 0$ situated entirely in K. Let $\varphi(r) \in C_0^\infty(0,\rho)$, $\varphi \neq 0$. Consider the mapping $\xi : K \to K$, having, in the polar coordinates, the form $(r,\varphi) \to (r, \varphi + N\psi(r))$, where N is a large parameter. $\xi(x) = x$ in a neighbourhood of ∂B, and we may continue it, setting $\xi(x) = x$ for all $x \in K \setminus B$. Choose the values $0 < r_1 < r_2 < r_3 < r_4 < \rho$, such that $\psi'(r) > 0$ on the segment $[r_1, r_4]$. Denote by G_i the annular domain $r_i < r < r_{i+1}$ $(i = 1,2,3)$. Let $\xi_t \subset \mathcal{D}$ be arbitrary flow, connecting Id and ξ. For $x, y \in K$, let us denote

$$\omega(x,y) = \int_0^1 d\arg\left(\xi_t(y) - \xi_t(x)\right) .$$

The following fact is evident:

LEMMA 2.5. $\inf\limits_{x,y \in G_3} \omega(x,y) - \sup\limits_{x,y \in G_2} \omega(x,y) = N(\psi(r_3) - \psi(r_2))$.

Hence, for at least one of the domains G_i $(i = 1,3)$

$$\left|\omega(x,y)\right| > N\frac{\psi(r_3) - \psi(r_1)}{2}$$

for all $x, y \in G_i$.

By Theorem 2.3, $J\{\xi_t\}_0^1 > CN^{1/2}$, for each path ξ_t connecting Id and ξ. N is unbounded and hence $\operatorname{diam}\mathcal{D} = \infty$.

2.4 Unattainable diffeomorphisms. In [S2], it was proved that if $\nu \geq 3$ then for each $\xi \in \mathcal{D}$ there exists a path ξ_t, connecting Id and ξ, such that $J\{\xi_t\}_0^1 < \infty$. Here we demonstrate that this is not true for $\nu = 2$.

THEOREM 2.6. *If $\nu = 2$, then there exists an unattainable diffeomorphism $\xi \in \mathcal{D}$. Moreover, this diffeomorphism may be chosen continuous up to ∂K, and fixed on ∂K.*

Proof: Choose a sequence of nonintersecting circles $B_1, B_2, \ldots$, in K, with centers $x_1, x_2, \ldots$, and radii $\rho_i, \rho_2, \ldots$, such that the points x_i converge to ∂K. Let ξ be fixed outside $\bigcup_{i=1}^\infty B_i$, and let $\xi|_{B_i}$ be a switch map, having in the polar coordinates, with the origin at x_i, the form

$$(r,\varphi) \to \left(r, \varphi + N_i\psi\left(\frac{r}{\rho_i}\right)\right) ;$$

if $\xi_t \subset \mathcal{D}$ is a flow, connecting Id and ξ, then

$$J\{\xi_t\}_0^1 \geq \sum_{i=1}^\infty J_i ,$$

where by Theorem 2.3

$$J_i = \int_0^1 dt \int_{B_i} \frac{1}{2} \left| \frac{\partial \xi_t(x)}{\partial x} \right|^2 dx \geq C\rho_i^2 N_i \ ;$$

if $J\{\xi_t\}_0^1 < \infty$, then $\sum_{i=1}^{\infty} J_i < \infty$. But we may choose $N_i > \rho_i^{-2}$, and then $\sum_{i=1}^{\infty} J_i = \infty$. Thus ξ is unattainable.

Note: The result of this section may be obtained using the methods of [ElR] as well.

2.5 Once more about the nonexistence of the shortest way in $\mathcal{D}(K)$ for $\nu \geq 3$. The GF may simplify and clarify the proof of the main result of [S1].

THEOREM 2.7. *Let $\nu \geq 3$. Then there exists an element $\xi \in \mathcal{D}$, such that there is no shortest way in $\mathcal{D}$ connecting* Id *and ξ. This means that for each way ξ_t, $0 \leq t \leq 1$, $\xi_0 =$ Id, $\xi_1 = \xi$, there exists another way $\xi_t' \subset \mathcal{D}$, connecting* Id *and ξ and such that $J\{\xi_t'\}_0^1 < J\{\xi_t\}_0^1$.*

Let us describe the construction of ξ and the proof of Theorem 2.7 for $\nu = 3$.

K is defined by the inequalities $|x_i| < \frac{1}{2}$, $i = 1, 2, 3$. Let us consider $\xi \in \mathcal{D}$ having the form $\xi(x_1, x_2, x_3) = (h(x_1, x_2), x_3)$, where $h \in \mathcal{D}(K^2)$.

THEOREM 2.8. *If*

$$\operatorname{dist}_{\mathcal{D}(K^3)}(\mathrm{Id}, \xi) < \operatorname{dist}_{\mathcal{D}(K^2)}(\mathrm{Id}, h) \ ,$$

then there is no shortest way in $\mathcal{D}(K^3)$ connecting Id *and ξ.*

Proof: First of all, it has been proved above that $\operatorname{dist}_{\mathcal{D}(K^3)}(\mathrm{Id}, \xi) \leq 2$; on the other hand it was just demonstrated that $\operatorname{dist}_{\mathcal{D}(K^2)}(\mathrm{Id}, h)$ may be arbitrarily large (and even infinite). Hence, there exist $h \in \mathcal{D}(K^2)$, satisfying the conditions of Theorem 2.8.

Consider arbitrary flow $\xi_t \subset \mathcal{D}(K_3)$, connecting Id and ξ, and construct the shorter one. Let $\xi_t^i(x)$ be the i-th coordinate of the point $\xi_t(x)$. Let

$$J_h\{\xi_t\}_0^1 = \int_0^1 dt \int_K \tfrac{1}{2} [(\dot{\xi}_t^1)^2 + (\dot{\xi}_t^2)^2] dx \ ,$$

$$J_v\{\xi_t\}_0^1 = \int_0^1 dt \int_K \tfrac{1}{2} [(\dot{\xi}_t^3)^2 dx \ ,$$

by the vertical and horizontal components of the action. Now, we construct the GF μ, connecting Id and ξ. The space $\Omega = K \times [0, 1]$. If $y \in K$, $z \in [0, 1]$,

$(y,z) \in \Omega$, then the path $x(y,z;t) := (\xi_t^1(y), \xi_t^2(y), z)$ is the projection of the trajectory $\xi_t(x)$ on the plane $x_3 = z$. From the form of ξ we see that μ is an incompressible GF, connecting Id and ξ, and $J\{\mu\}_0^1 = J_h\{\xi_t\}_0^1$.

If $J_v\{\xi_t\}_0^1 > 0$, then $J\{\mu\}_0^1 < J\{\xi_t\}_0^1$. By the Approximation Theorem, there exists a smooth flow ξ_t', connecting Id and ξ, such that ξ' approximates μ together with J, so that $|J\{\xi_t'\}_0^1 - J\{\xi_t\}_0^1| < \frac{1}{2}J_v\{\xi_t\}_0^1$; hence, $J\{\xi_t'\}_0^1 < J\{\xi_t\}_0^1$.

If $J_v\{\xi_t\}_0^1 = 0$, then $\dot{\xi}_t^3 = 0$ a.e. in K, each horizontal section $x_3 = z$ is invariant under ξ_t, and we have a family of flows ζ_t^z in these sections, connecting Id and h. By our hypothesis $J\{\xi_t^z\}_0^1 > \inf_{\xi_t \subset \mathcal{D}(K^3)} J\{\xi_t\}_0^1$ and, therefore, ξ_t cannot be a minimal path connecting Id and ξ. This contradiction proves the theorem.

2.6 Conjugate points in $\mathcal{D}$. Let $\mathcal{M}$ be a Riemannian manifold, and let $\gamma : t \to x(t) \subset \mathcal{M}$, $0 \le t \le T$, be a geodesic in $\mathcal{M}$. A point $x(t_0)$ is the first conjugate to $x(0)$ along the geodesic γ if, for each $t_1 < t_0$, the piece of γ connecting $x(0)$ and $x(t_1)$ has the least length (and action) among all close paths, connecting the points $x(0)$ and $x(t)$, and for $t_1 > t_0$ this segment fails to be locally the shortest: i.e. for each $\varepsilon > 0$, there exists a path $y(t) \subset \mathcal{M}$, connecting $x(0)$ and $x(t_1)$ such that $\mathrm{dist}(x(t), y(t)) < \varepsilon$ for all t, $0 \le t \le t_1$, and $L\{y(t)\}_0^{t_1} < L\{x(t)\}_0^{t_1}$. This is one of the possible definitions of conjugate point; all of them are equivalent for finite-dimensional Riemannian manifolds, but they split into different definitions for infinite dimensional ones (see [G]).

The problem of existence of conjugate points on the manifold $\mathcal{D}$ was posed by Arnold in 1966 ([Ar]). Recently G. Misiolek proved the existence of conjugate points on $\mathcal{D}(G)$, where G is a flat 2-dimensional torus ([Mi]). This construction is likely to generalize on the general case of $\mathcal{D}(\mathcal{M})$ for arbitrary 2-dimensional compact manifold $\mathcal{M}$. Misiolek also conjectured (and proved for a flat torus) that for each 2-dimensional manifold there exist geodesics without conjugate points.

But for $\nu \ge 3$, the situation is quite different: the conjugate points become *indispensible*: on each sufficiently long arc of geodesic in $\mathcal{D}(G)$ there exist conjugate points. More precisely, the following assertion is true.

THEOREM 2.9. *Let $G = K^\nu$ be a unit ν-dimensional cube, $\nu \ge 3$. Let $\xi \subset \mathcal{D}(G)$, $0 \le t \le T$, be an arbitrarily piecewise-smooth curve such that its length $L\{\xi_t\}_0^T$ is more than $\mathrm{diam}\,\mathcal{D}(G)$. Then ξ_t is not locally the shortest path connecting ξ_0 and ξ_T. This means that for each $\varepsilon > 0$ there exist a smooth path $\eta_t \subset \mathcal{D}$, $0 \le t \le T$, $\eta_0 = \xi_0$, $\eta_T = \xi_T$, such that $\mathrm{dist}(\xi_t, \eta_t) < \varepsilon$ for all $t \in [0,T]$, and $L\{\eta_t\}_0^T < L\{\xi_t\}_0^T$.*

In particular, this means that on each geodesic line longer than $\mathrm{diam}\,\mathcal{D}$

there exist conjugate points.

Proof: If $L\{\zeta_t\}_0^T > \operatorname{diam}\mathcal{D} = \sup_{\xi\eta} \operatorname{dist}_{\mathcal{D}}(\xi,\eta)$, then there exists a path ζ_t, $0 \le t \le T$, connecting ξ_0 and ξ_T, such that $J\{\zeta_t\}_0^T < J\{\xi_t\}_0^T$. Let $\mu_{\xi_t}, \mu_{\zeta_t}$ be corresponding GF's. Let us consider the mixture of these GF's, $\nu = (1-\delta)\mu_{\xi_t} + \delta\mu_{\zeta_t}$, $0 \le \delta \le 1$. This is a GF connecting ξ_0 and ξ_T, and its action

$$J\{\nu\}_0^T = (1-\delta)J\{\mu_{\xi_t}\}_0^T + \delta J\{\mu_{\zeta_t}\}_0^1 < J\{\mu_{\xi_t}\}_0^T \ .$$

If $\delta \to 0$, then $\nu \rightharpoonup \mu_{\xi_t}$ weak $*$. For each δ, by the Approximation Theorem, we may find a smooth flow η_t, s.t. it is arbitrarily close (in the weak $*$ sense) to the GF ν, and $J\{\eta_t\}_0^T$ is a arbitrarily close to $J\{\nu\}_0^T$. Hence, we may find η_t such that $J\{\eta_t\}_0^T < J\{\xi_t\}_0^T$. But the flow η_t is $w*$-close to ξ_t. By Theorem 2.2, this means precisely that, for each finite sequence $0 \le t_1 \le t_2 < \ldots < t_N = T$, and for each $\varepsilon > 0$, we may find δ and a smooth flow η_t approximating $\nu = (1-\delta)\mu_{\eta_t} + \delta\mu_{\zeta_t}$, such that $\operatorname{mes}\{x \mid |\eta_{t_i}(x) - \zeta_{t_i}(x)| > \varepsilon\} < \varepsilon$ for all $i = 1,\ldots,N$. For each other time moment t_i, $t_i < t < t_{i+1}$,

$$\begin{aligned}
\|\xi_t - \eta_t\|_{L^2} &\le \|\xi_{t_i} - \eta_{t_i}\|_{L^2} + (t-t_i)\left[\left(\frac{2J\{\xi_t\}_0^T}{(t-t_i)}\right)^{1/2} + \left(\frac{2J\{\zeta_t\}_0^T}{(t-t_i)}\right)^{1/2}\right] \\
&\le \|\xi_{t_i} - \eta_{t_i}\|_{L^2} + C(t-t_i)^{1/2} \le \\
&\le \|\xi_{t_i} - \eta_{t_i}\|_{L^2} + C\cdot\max_j (t_{j+1}-t_j)^{1/2} \ ,
\end{aligned}$$

and for $\max(t_{i+1}-t_i)$ sufficiently small,

$$\|\zeta_t - \eta_t\|_i < 2\varepsilon$$

for all $t \in [0,T]$. By Theorem 2.2, this means that

$$\operatorname{dist}_{\mathcal{D}}(\zeta_t,\eta_t) < C\varepsilon^\alpha \ , \qquad \alpha > 0 \ ,$$

and is small together with ε. Thus, for each $\varepsilon > 0$, we have constructed a path η_t, such that $J\{\eta_t\}_0^T < J\{\xi_t\}_0^T$, $\eta_0 = \xi_0$ $\eta_T = \xi_T$, and $\operatorname{dist}_{\mathcal{D}}(\xi_t,\eta_t) < C\varepsilon^\alpha$. This concludes the proof.

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Submitted: October 1993