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ATTAINABLE DIFFEOMORPHISMS

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Introduction

Let K be a bounded domain in $\mathbb{R}^\nu$ ($\nu = 2, 3, \ldots$); let ξ be a diffeomorphism of K onto itself, preserving orientation and volume. ξ is called *attainable*, if there exists a one-parameter family of volume preserving diffeomorphisms $\xi_t : K \to K$, $0 \le t \le 1$, such that $\xi_0 = Id =$ the identity mapping, $\xi_1 = \xi$, and the action of the flow ξ_t is finite:

$$J\{\xi_t\}_{t=0}^1 = \int_0^1 \int_K \left|\frac{\partial \xi_t(x)}{\partial t}\right|^2 dx\, dt < \infty .$$

The set of all attainable diffeomorphisms is denoted by $\mathcal{D}(K) = \mathcal{D}$. It is a natural configuration space of a classical ideal incompressible fluid. In fact, each position or configuration of the fluid is determined by positions of all its particles. Any two configurations differ by some permutation of fluid particles. In classical hydrodynamics these permutations are assumed to be orientation and volume preserving diffeomorphisms of the flow domain (see [EM]). If we fix some initial position of the fluid, then we may identify all other positions with corresponding diffeomorphisms. Only attainable diffeomorphisms are physically reasonable, otherwise it would be impossible for the fluid to reach them (requiring infinite energy and/or time).

The possible existence of non-attainable diffeomorphisms is connected with arbitrarily complicated behaviour of the diffeomorphism near the boundary. However, it was conjectured in [S], that if the dimension $\nu \ge 3$, and the domain K is simply connected, then every orientation and volume preserving diffeomorphism of K is attainable, i.e. $\mathcal{D}$ coincides with the group of all such diffeomorphisms.

In this work we prove this conjecture. Our main result is

THEOREM A. *Let K be a unit cube in $\mathbb{R}^\nu$, $\nu \ge 3$. Then every orientation and volume preserving diffeomorphism $\xi : K \to K$ is attainable.*

One of the main results of [S] is the following: if $\nu \ge 3$, then there is a constant C, depending only on ν, such that for each two attainable diffeomorphisms $\xi, \eta \in \mathcal{D}$ there exist a path $\xi_t \subset \mathcal{D}$, $0 \le t \le 1$, such that $\xi_0 = \xi$, $\xi_1 = \eta$, and the action $J\{\xi_t\}_{t=0}^1 < C$; we may say, that the diameter of the configuration space $\mathcal{D}$ is finite. But the scope of this result has remained unclear, because there was no good description of the set $\mathcal{D}$ of all attainable diffeomorphisms. Now we see that our result is valid for the whole group of volume-preserving diffeomorphisms of K; in particular we may now assert that $\mathcal{D}$ is a group, which was not clear *a priori* (for example, if $\xi_t \in \mathcal{D}$, $0 \le t \le 1$, $\xi_0 = Id$, $\xi_1 = \xi$, and $J\{\xi_t^1\}_{t=0}^1 < \infty$, then it is not always true that $J\{\xi_t^{-1}\}_{t=0}^1 < \infty$; so it is not so clear that $\xi^{-1} \in \mathcal{D}$).

Theorem A is not implied directly by the finiteness of the diameter of $\mathcal{D}$; however, the proofs of these results are similar, so that we have to reproduce here most of the steps of the previous work [S] for a slightly different situation.

For simplicity, we restrict ourself to the case $\nu = 3$; moreover, most of the steps of the proof are done equally or with minor modifications for all $\nu \ge 2$, so we'll describe them for $\nu = 2$. The steps that are identical with corresponding steps in [S] are only sketched.

It was also conjectured in [S] that in the case $\nu = 2$ there exist unattainable diffeomorphisms; this will be proved elsewhere.

1. Decomposition of a Diffeomorphism

Let us denote by $\mathcal{SD} = \mathcal{SD}(K)$ the group of all orientation and measure preserving diffeomorphisms of K (it will occur below, that $\mathcal{SD} = \mathcal{D}$, but before then these sets must be regarded as different). Our proof is based on the following theorem, asserting the pathwise connectedness of the group $\mathcal{SD}$ in compact-open C^∞-topology and proved in [M, Lemma 5].

THEOREM 1.1. *For each diffeomorphism $\xi \in \mathcal{SD}$ there exists a one-parameter family $\xi_t \in \mathcal{SD}$, $0 \le t \le 1$, such that ξ_t depends continuously and piecewise smoothly on t in each subdomain $K' \subset K$, $\overline{K}' \subset K$, $\xi_0 = Id$, $\xi_1 = \xi$.*

We also use the following two lemmas, proved in [S]:

LEMMA 1.2. *Let B be a unit ball in $\mathbb{R}^\nu$. There exists a smooth mapping $\rho : B \to K$, such that*
(i) *its Jacobian $\left|\frac{\partial \rho(x)}{\partial x}\right| \equiv$ const in B;*
(ii) *its first derivatives are bounded in all of B;*

(iii) *ρ preserves the symmetry of K.*

LEMMA 1.3. *For each $r > 0$, let $B_r \subset \mathbb{R}^\nu$ be a ball of radius r; $0 < r_1 < r_2$. Let $\mathcal{H}_1$ be the space of all smooth divergence-free vector fields in B_{r_2}, and $\mathcal{H}_2 \subset \mathcal{H}_1$ the space of all smooth divergence free vector fields $u(x)$, such that* supp *$u(x) \subset B_{r_2}$. Then there exists a linear operator $\mathcal{R} : \mathcal{H}_1 \to \mathcal{H}_2$, continuous in C^∞, such that $\mathcal{R}u(x) \equiv u(x)$ in B_{r_1} for all $u(x) \in \mathcal{H}_1$.*

Let $\xi \in \mathcal{D}$; a set $K' \subset K$ is called a support of ξ ($K' = \operatorname{supp}\xi$) if K' is the least set, such that $\xi = Id$ in $K \backslash K'$. We say that diffeomorphism $\xi \in \mathcal{D}(K')$, if $\operatorname{supp}\xi \subset K'$, and there exists a path $\xi_t \subset \mathcal{D}$, such that $\xi_0 = Id$, $\operatorname{supp}\xi_t \subset K'$, and $J\{\xi_t\}_{t=0}^1 < \infty$. For $0 < \varepsilon < \delta < \frac{1}{2}$ we define an annulus domain $K_{\varepsilon,\delta} = \{x \in K \mid \varepsilon < \operatorname{dist}(x, \partial K) < \delta\}$. The first step of the proof of Theorem A is the following

LEMMA 1.4. (i) *Let $\xi \in \mathcal{SD}$; then for each sequence $\delta_k \to 0$ there exist a sequence $\varepsilon_k \to 0$, and a factorization*

$$\xi = \ldots \circ \xi_3 \circ \xi_2 \circ \xi_1 \ ,$$

such that all $\xi_k \in \mathcal{D}$, $\operatorname{supp}\xi_k \subset K_{\xi_k,\delta_k}$, and for each $x \in K$ there exists $m = m(x)$, such that $\xi(x) = \xi_m \circ \ldots \circ \xi_1$ (i.e. the sequence of partial products stabilizes).

(ii) *Let $\xi_t \subset \mathcal{SD}$, $0 \le t \le 1$, be a family of diffeomorphisms, depending smoothly on t, $\xi_0 = Id$, $\xi_1 = \xi$. Then for each sequence $\delta_k \to 0$ there exist a sequence $\varepsilon_{k,t} \to 0$ and a factorization*

$$\xi_t = \ldots \circ \xi_{3,t} \circ \xi_{2,t} \circ \xi_{1,t} \ ,$$

such that $\xi_{k,t}$ is a smooth path in $\mathcal{D}$ for all k; $\operatorname{supp}\xi_{k,t} \subset K_{\varepsilon_k,\delta_k}$; $\xi_{k,0} = Id$; the sequence of partial products $\xi_{m,t} \circ \ldots \circ \xi_{2,t} \circ \xi_{1,t}(x)$ stabilizes for each $x \in K$.

Proof: (i) is a particular case of (ii), so we shall prove (ii). Let ξ_t be the flow connecting Id and ξ in $\mathcal{SD}$; let $v_t(y) = \dot{\xi}_t \circ \xi_t^{-1}(y)$, $y \in K$, be the Euler velocity field of this flow. Choose a sequence $\varepsilon_k \searrow 0$. Using Lemmas 1.2 and 1.3, we can find incompressible vector fields $v_{k,t}(y)$ such that (a) $v_{k,t}(y) \equiv v(y)$, if dist $(y, \partial K) \ge 2\varepsilon_k$; (b) $v_{k,t}(y) \equiv 0$, if dist $(y, \partial K) \le \varepsilon_k$. Let us construct the flows $\tilde{\xi}_{k,t} : K \to K$, integrating the fields $v_{k,t}$,

$$\frac{\partial \tilde{\xi}_{k,t}(x)}{\partial t} = v_{k,t} \circ \tilde{\xi}_{k,t}(x), \quad \tilde{\xi}_{k,0}(x) = x, \quad 0 \le t \le 1 \ .$$

It is easy to see that $\tilde{\xi}_{k,t} = Id$ in the ε_k-neighbourhood of ∂K and that there exists a sequence $\delta'_k \to 0$ such that $\tilde{\xi}_{k,t}$ coincides with ξ_t outside the δ'_k-neighbourhood of ∂K. We may choose a subsequence, and assume that $\delta'_k < \delta_k$.

Now, we may set $\xi_t = \ldots \circ \xi_{2,t} \circ \xi_{1,t}$, where $\xi_{1,t} = \tilde{\xi}_{1,t}$, $\xi_{2,t} = \tilde{\xi}_{2,t} \circ \tilde{\xi}^{-1}_{1,t}, \ldots, \xi_{i,t} = \tilde{\xi}_{i,t} \circ \tilde{\xi}^{-1}_{i-1,t}$.

This is the required decomposition: for each $x \in K$ the sequence of partial products $\xi_{m,t} \circ \ldots \circ \xi_{1,t}(x)$ stabilizes for $m > m(x)$ such that $\delta_{m(x)} < \operatorname{dist}(x, \partial K)$. Q.E.D.

Now we approach the main point of the proof of Theorem A.

LEMMA 1.5. *Let $\nu \geq 3$, $0 < \varepsilon < \delta$, and $\xi \in \mathcal{D}(K_{\varepsilon,\delta})$. Then there exists a flow ξ_t, $0 \leq t \leq 1$, $\xi_0 = Id$, $\xi_1 = \xi$, such that*

(i) *$\operatorname{supp} \xi_t \subset K_{\delta,\varepsilon}$ for all t, $0 \leq t \leq 1$;*

(ii) *$J\{\xi_t\}_{t=0}^1 = \int_0^1 \int_K \left| \frac{\partial}{\partial t} \xi_t(x) \right|^2 dx\, dt \leq C \cdot (\varepsilon - \delta)$, where C depends only on ν.*

Most of the rest of this work is devoted to proving the above lemma.

Proof of Theorem A: Let $\xi \in \mathcal{SD}$ and $\delta_k = 2^{-2k}$. By Lemma 1.4, there exist positive $\varepsilon_k < \delta_k$, and mappings ξ_k such that $\operatorname{supp} \xi_k \subset K_{\varepsilon_k,\delta_k}$ and $\xi = \ldots \circ \xi_2 \circ \xi_1$. Lemma 1.5 asserts that there exist flows $\xi_{k,t}$, $0 \leq t \leq 1$, such that $\operatorname{supp} \xi_{k,t} \subset K_{\varepsilon_k,\delta_k}$, $\xi_{k,0} = Id$, $\xi_{k,1} = \xi_k$ and $J\{\xi_{k,t}\} \leq C \cdot 2^{-2k}$. Let $t_k = 1 - 2^{-k}$; we define the following flow in K:

$$\eta_t = \begin{cases} \xi_{1,2t}, & 0 \leq t \leq t_1; \\ \xi_{2,2^2(t-t_1)} \circ \xi_{1,1}, & t_1 \leq t \leq t_2; \\ \xi_{3,2^3(t-t_2)} \circ \xi_{2,1} \circ \xi_{1,1}, & t_2 \leq t \leq t_3; \\ \cdots\cdots\cdots & \\ \xi_{k,2^k(t-t_k)} \circ \xi_{k-1,1} \circ \ldots \circ \xi_{1,1}, & t_{k-1} \leq t \leq t_k; \\ \cdots\cdots\cdots & \end{cases}$$

This flow is piecewise smooth in t, and for each $x \in K$ there is k such that, for all $t > t_k$, $\eta_t(x) = \xi(x)$. In order to estimate the action $J\{\eta_t\}_{t=0}^1$, we shall note that, for each flow $\zeta_t \subset \mathcal{D}$, $0 \leq t \leq 1$ and each $a > 0$,

$$J\{\zeta_{at}\}_{t=0}^{a^{-1}} = a \cdot J\{\zeta_t\}_{t=0}^1 .$$

Hence,

$$J\{\eta_t\}_{t=0}^1 = 2J\{\xi_{1,t}\}_{t=0}^1 + 2^2 J\{\xi_{2,t}\}_{t=0}^1 + \ldots = C \cdot (2 \cdot 2^{-2} + 2^2 \cdot 2^{-4} + \ldots) = C \cdot$$

Thus Theorem A is proved.

The rest of the work is devoted to the proof of Lemma 1.5.

2. Reduction to the Discrete Configuration

In this section we shall construct an isotopy, reducing a diffeomorphism $\xi \in \mathcal{D}(K_{\varepsilon,\delta})$ to a permutation of small cubes. All our considerations apply equally, or with obvious modifications, for all $\nu \geq 3$. We shall, therefore, confine ourselves to the case $\nu = 3$.

First we recall some notions from [S]. Let N be a (large) integer; it is convenient to assume that N is a power of 2. Let us divide K into N^ν equal cubes. Let K_N be the set of these cubes; elements of K_N will be denoted by κ. We shall call each permutation $\sigma : K_N \to K_N$ a *discrete configuration*; the set of all configurations will be denoted by $\mathcal{D}_N$. Two cubes κ, κ' are called *neighbours* if they have a common $(\nu - 1)$-dimensional face. Permutation $\sigma \in D_N$ is called *elementary* if, for each $\kappa \in K_N$, either $\sigma\kappa = \kappa$ or $\sigma\kappa$ and κ are neighbours, and $\sigma^2 = Id$. A sequence $\tau_1, \tau_2, \dots, \tau_k$ of elementary permutations is called a *discrete flow*, k is its *duration*. We say, that the discrete flow $\tau_1, \dots, \tau_k$ transforms configuration σ into configuration σ' $(\sigma, \sigma' \in D_N)$, if $\sigma' = \tau_k \circ \tau_{k-1} \circ \dots \circ \tau_1 \circ \sigma$.

(It should be explained that we identify discrete configurations of the set of cubes κ with permutations because if we can distinguish between different cubes κ, and fix some initial position for each of them, then each other configuration of the set of cubes κ differs from the initial one by some uniquely determined permutation $\sigma \in \mathcal{D}_N$, just as for a real fluid.)

Let $K'_N \subseteq K_N$; a subgroup of permutations $\sigma \in \mathcal{D}_N$, fixed outside K'_N, is denoted by $\mathcal{D}_N(K'_N)$, $\mathcal{D}_N(K_N) = \mathcal{D}_N$.

A measurable mapping $\xi : K \to K$ is called a *discontinuous configuration* if (a) ξ is smooth and volume-preserving in each cube $\kappa \in K_N$ and continuous in $\overline{\kappa}$, and (b) ξ is almost everywhere one-to-one. The set of all discontinuous configurations is denoted by $\mathcal{F}_N$. In particular, each permutation $\sigma \in \mathcal{D}_N$ may be regarded as an element of $\mathcal{F}_N$, so we have a natural embedding $\mathcal{D}_N \to \mathcal{F}_N$.

A one-parameter family $\xi_t \subset \mathcal{F}_N$, $0 \leq t \leq 1$, is called a discontinuous flow, for which we may define an action

$$J\{\xi_t\}_{t=0}^1 = \int_0^1 \int_K \left|\dot{\xi}_t(x)\right|^2 dx\, dt\ .$$

If $K'_N \subseteq K_N$, then $\mathcal{F}_N(K'_N)$ is a subset of $\mathcal{F}_N$, consisting of all discontinuous configurations ξ, such that $\xi = Id$ outside $\bigcup_{\kappa \in K'_N} \kappa$.

We begin with the approximation of smooth volume preserving diffeomorphisms by permutations of small cubes.

It was proved in [S] that *for each diffeomorphism* $\xi \in \mathcal{D}$, *fixed near* ∂K, *and for each* $\varepsilon > 0$ *there exist an integer* N *and a path* $\xi_t \subset \mathcal{F}_N$, $0 \le t \le 1$, $\xi_0 = \xi$, *such that* $\xi_1 \in \mathcal{D}_N$, *and* $J\{\xi_t\}_{t=0}^1 < \varepsilon$.

In this work we shall use the following modification of this result.

LEMMA 2.1. *Let* $0 < \delta_1 < \delta_2$, *and* $K' = K_{\delta_1,\delta_2}$; *let* $\xi \in \mathcal{D}(K')$, i.e. $\xi \in \mathcal{D}$, $\operatorname{supp}\xi \subset K'$, *and there exists a path* $\xi_t \subset \mathcal{D}$, $0 \le t \le 1$, *such that* $\xi_0 = Id$, $\xi_1 = \xi$, $\operatorname{supp}\xi_t \subset K'$ *for all* t, *and* $J\{\xi_t\}_{t=0}^1 < \infty$. *Then for each* $\varepsilon > 0$ *there exist* $N > 0$ *and a path* $\xi_t \subset \mathcal{F}_N(K'_N)$, $1 \le t \le 2$, *such that* $\xi_1 = \xi$, $\xi_2 \in \mathcal{D}_N(K'_N)$, *and* $J\{\xi_t\}_{t=1}^2 < \varepsilon$.

(Here $K'_N = \{\kappa \in K_N \mid \kappa \cap K' \neq \emptyset\}$.)

Note: P. Lax in 1971 [L] proved that each measure preserving homeomorphism $f : \overline{K} \to \overline{K}$ may be approximated by some permutation of small cubes. In this work a stronger result is established, that there is an isotopy, connecting f and some permutation and having arbitrarily small action. This assertion is not implied directly by Lax's theorem.

Proof: The proof is almost identical to the proof of Lemma 4.2 in [S]. As a first step, we represent $\xi \in \mathcal{D}(K')$ as a composition of mappings such that each of them differs slightly from Id. Let ξ_t, $0 \le t \le 1$, be the path in $\mathcal{D}(K')$ connecting Id and ξ. We may set

$$\xi = \eta_k \circ \eta_{k-1} \circ \ldots \circ \eta_1 \ ,$$

where

$$\eta_i = \xi_{i/k} \circ \xi^{-1}_{(i-1)/k} \ , \quad i = 1, \ldots, k \ .$$

If k is large enough, then all η_i are close to Id in C^1.

LEMMA 2.2. *Suppose that all the conditions of Lemma 2.1 are fulfilled, and that the mapping* $\xi \in \mathcal{D}(K')$ *is* C^1*-close to Id. Then the conclusion of Lemma 2.1 is true.*

We shall prove this lemma a little later, and now finish the proof of Lemma 2.1. Lemma 2.2 asserts, that for each $i \le k$, and each $\varepsilon > 0$ there are N_i and a path $\eta_{i,s}$ in $\mathcal{F}_{N_i}(K')$, $0 \le s \le 1$, such that $J\{\eta_{i,s}\}_{s=0}^1 < \varepsilon$. Let us define a composite path

$$\xi_{1+t} = \begin{cases} \eta_k \circ \eta_{k-1} \circ \ldots \circ \eta_2 \circ \eta_{1,t\cdot k}, & 0 \le t < \frac{1}{k}; \\ \eta_k \circ \eta_{k-1} \circ \ldots \circ \eta_{2,t\cdot k-1} \circ \eta_{1,1}, & \frac{1}{k} \le t < \frac{2}{k}; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \eta_k \circ \eta_{k-1} \circ \ldots \circ \eta_{i,t\cdot k-i+1} \circ \eta_{i-1,1} \circ \ldots \circ \eta_{1,1}, & \frac{i-1}{k} \le t < \frac{i}{k}; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \eta_{k,t\cdot k-k+1} \circ \eta_{k-1,1} \circ \ldots \circ \eta_{1,1}, & \frac{k-1}{k} \le t \le 1 . \end{cases}$$

If $J_i = J\{\eta_{i,s}\}_{s=0}^1$, and $A_i = \sup_{x\in K} \|\frac{\partial \eta_i(x)}{\partial x}\|$, where $\|\cdot\|$ denotes the matrix (operator) norm, then

$$\begin{aligned} J\{\xi_t\}_{t=1}^2 &\le k[A_k \cdot A_{k-1} \cdot \ldots \cdot A_2 \cdot J_1 + A_k \cdot A_{k-1} \cdot \ldots A_3 \cdot J_2 + \ldots + J_k] \\ &\le k \cdot A_k \cdot A_{k-1} \cdot \ldots \cdot A_2(J_1 + J_2 + \ldots + J_k) . \end{aligned}$$

By Lemma 2.2, J_i may be done arbitrarily small, if we take N_i sufficiently large and choose appropriate paths in $\mathcal{F}_{N_i}(K')$. If all N_i are powers of 2, then the path ξ_t is in $\mathcal{F}_N(K')$, $N = \max_i N_i$, and $\xi_2 \in \mathcal{D}_N(K'_N)$. Thus Lemma 2.1 is proved.

Our proof of Lemma 2.2 is based on the following Lemmas 2.3 and 2.4. Let K be a unit cube $0 \le x_i \le 1$, $i, \ldots, \nu$. Consider the family of surfaces in K, defined as $x_n = \varphi(x', a)$, $x' = (x_1, \ldots, x_{\nu-1})$, $0 \le a \le 1$, such that (a) $\varphi(x', 0) = 0$, (b) $\varphi(x', 1) = 1$, (c) $C^{-1} < \frac{\partial\varphi}{\partial a} < C$, (d) $|\partial\varphi/\partial x'| < C$, (e) $\int \varphi(x', a)dx' = \overline{\varphi}(a)$.

LEMMA 2.3. *There exists an isotopy $f_t \subset \mathcal{D}(K)$, $0 \le t \le 1$, such that $f_0 = Id$, f_1 transforms the surfaces $x_\nu = \varphi(x', a)$ into the surfaces $x_\nu = \overline{\varphi}(a) = \int \varphi(x, a')dx'$ and $J\{f_t\}_{t=0}^1 < C'$ where C' depends on C only.*

Proof: Let us begin with the case $\nu = 2$. Let $\varphi_t(x', a) = t \cdot \overline{\varphi}(a) + (1 - t)\varphi(x', a)$. For each t, the mapping f_t transforming the line $x_\nu = \varphi(x', a)$ into the line $x_\nu = \varphi(x', a)$, and area-preserving, is given implicitly by the formulas

$$\begin{cases} f_t(x_1, x_2) = (y_1, y_2) ; \\ x_2 = \varphi_0(x_1, a) ; \\ \int_0^{x_1} \frac{\partial\varphi_0(z,a)}{\partial a} dz = \int_0^{y_1} \frac{\partial\varphi_t(z,a)}{\partial a} dz ; \\ y_2 = \varphi_t(y_1, a) . \end{cases}$$

It is easy to check that this family satisfies all the requirements of the lemma for $\nu = 2$. For $\nu = 3$ we shall first construct, in each 2-dimensional section of K by the plane $x_2 = \text{const}$, an area-preserving flow $f^1_{x_2,t}(x_1, x_3)$, $0 \le t \le 1$, such that $f^1_{x_2,0} = Id$, and $f_{x_2,1}$ transforms the lines of the form $x_3 = \varphi(x_1, x_2, a)$ into the lines $x_3 = \overline{\varphi}(x_2, a) = \int_0^1 \varphi(x_1, x_2, a)dx_1$. This construction is done just as described before for $\nu = 2$.

For $t = 1$, we get a family of cylindrical surfaces $x_3 = \overline{\varphi}(x_2, a)$, and the same construction independent of x_1 gives us an isotopy f^2_t, $0 \le t \le 1$, such that f^2_1 transforms these surfaces into the plane $x_3 = \overline{\varphi}(a)$. Then

$$f_t = \begin{cases} f^1_{2t}\,, & 0 \le t \le \frac{1}{2} \\ f^2_{2t-2} \circ f^1_1\,, & \frac{1}{2} \le t \le 1 \end{cases}$$

is the required isotopy.

For higher ν the construction is similar. The action estimate is evident.

LEMMA 2.4. *Consider in K p surfaces of the form $x_\nu = g_i(x')$, $i = 1, \dots, p$, $x_\nu = g_i(x') \in C^\infty$, $0 \le g_i(x') \le 1$, and suppose that $C^{-1}p^{-1} < g_{i+1}(x') - g_i(x') < Cp^{-1}$, $\left|\frac{\partial g_i(x')}{\partial x'}\right| < C$ for all i, x'. Then there exists an isotopy $f_t \subset \mathcal{D}(K)$, $0 \le t \le 1$, such that $f_0 = Id$, f_1 transforms these surfaces into the surfaces $x_\nu = \overline{g}_i = \int g_i(x')dx'$, and $J\{f_t\}^1_{t=0} < C'$, where C' depends only on C.*

Proof: There exists an interpolating function $\varphi(x', a)$, such that $\varphi(x', 0) \equiv 0$, $\varphi(x', 1) \equiv 1$, $C^{-1} < \frac{\partial \varphi}{\partial a} < C$, $\left|\frac{\partial \varphi}{\partial x'}\right| < C$, and $\varphi(x', \frac{i}{p}) \equiv g_i(x')$ (for example, $\varphi(x', a)$ may be some appropriate polynomial with respect to a). Then Lemma 2.3 may be applied to establish the required result.

Proof of Lemma 2.2: Let $\xi \in \mathcal{D}(K')$ be C^1-close to Id. We shall construct an isotopy, transforming ξ into some permutation of small cubes $\kappa \in K'_N$ having a small action. Our construction consists of a number of steps. We shall describe them for $\nu = 3$; generalization to higher ν is straightforward. We begin with the case $K' = K$.

Step 1. Choose $n \in \mathbb{Z}$ large enough; it is convenient to assume that n is a power of 2. Partition K into the "bars" K_{ij}, $\frac{i-1}{n} \le x_1 < \frac{i}{n}$, $\frac{j-1}{n} \le x_2 < \frac{j}{n}$ $(i, j = 1, \dots, n)$. According to Lemma 2.3, for each i, j there is an isotopy $f_{ijt} : K_{ij} \to K_{ij}$, $0 \le t \le 1$, transforming the surfaces $x_3 = \text{const}$ into the surfaces $\xi^3(x) = \text{const}$ (where $\xi(x) = (\xi^1(x), \dots, \xi^3(x))$).

Then, $\xi_t = \xi \circ f_{ijt} : K_{ij} \to \xi(K_{ij})$, $0 \le t \le 1$, is an isotopy of $\xi\big|_{K_{ij}}$ into a mapping ξ_1, such that it transforms the surface $x_3 = a$ in K_{ij} into the surface $x_3 = \varphi_{ij}(a)$ in $\xi(K_{ij})$. These isotopies should be done in all K_{ij} simultaneously. It is easy to see that $\xi_t \in \mathcal{F}_n$ for all t, and that the action $J\{\xi_t\}_{t=0}^1$ tends to 0 when $n \to \infty$.

Step 2. Let us partition K into the horizontal layers $K^p : \frac{p-1}{n} < x_3 < \frac{p}{n}$ $(p = 1, \dots, n)$. Then $K_{ij}^p = \xi_1^{-1}\big(\xi(K_{ij}) \cap K^p\big)$ is a parallelopiped, $K_{ij}^p = \{x \in K_{ij},\ b_{ij}^p \le x_3 < b_{ij}^{p+1}\}$.

Step 3. We may change the mapping ξ arbitrarily small, so that the coordinates of all the vertices of all K_{ij}^p are 2-rational.

Step 4. Consider domain $G_j^p = \bigcup_i \xi(K_{ij}^p) \subset K^p$. This is a curvilinear bar with two horizontal faces. According to Lemma 2.4, we may construct an isotopy $g_{jt}^p : G_j^p \to G_j^p$, $0 \le t \le 1$, such that g_{j1}^p transforms each interface between the cells $\xi_1(K_{ij}^p)$ and $\xi_1(K_{i+1,j}^p)$ into the surface $x_2 =$ const. $G_{ij}^p = g_{j1}^p \circ \xi_1(K_{ij}^p)$ is a curvilinear parallelopiped having 2 faces parallel to the plane $x_1 - x_2$, and 2 faces parallel to the plane $x_1 - x_3$. Let $g_t : K \to K$, $g_t\big|_{G_j^p} = g_{jt}^p$. It is easily seen, that $J\{g_t\}_{t=0}^1 \to 0$ when $n \to \infty$.

Step 5. Using Lemma 2.3, we construct an isotopy $h_t : G_{ij}^p \to G_{ij}^p$, $0 \le t \le 1$, such that $h_1 \circ g_1 \circ \xi_1$ transforms each surface $x_2 = c$ in K_{ij}^p into the surface $x_2 = \chi_{ij}^p(c)$ in G_{ij}^k.

Step 6. The faces of all G_{ij}^p, parallel to the plane $x_1 - x_3$, have equations $x_2 = c_{ij}^p$. Changing ξ arbitrarily small, we may assume all c_{ij}^p to be 2-rational. Suppose that 2^q is their common denominator. The planes $x_2 = \frac{\ell}{2^q}$ $(\ell = 0, \dots, 2^q)$ divide K into slices $K_\ell : \frac{\ell-1}{2^q} \le x_2 \le \frac{\ell}{2^q}$. Let $G_{ij}^k \cap K_\ell = H_{\ell j}^k$. Then $(h_1 \circ g_1 \circ \xi_1)^{-1}(H_{\ell j}^k) = Q_{\ell j}^k$ is a parellelopiped, for the planes $x_2 =$ const are transferred into the planes of the same type by the mapping $h_1 \circ g_1 \circ \xi_1$.

Step 7. By changing ξ arbitrarily small, we may make the coordinates of all the vertices of all $Q_{\ell j}^k$ 2-rational.

Step 8. Consider $\bigcup_j H_{\ell j}^k = P_\ell^k$. Applying Lemma 2.4 once more, we may construct an isotopy r_t, $0 \le t \le 1$, of the bar P_ℓ^k, making the interfaces between all the cells $H_{\ell j}^k$ and $H_{\ell,j+1}^k$ the planes parallel to the $x_2 - x_3$ plane. Denote it by r_t, $0 \le t \le 1$. The mapping $r_1 \circ h_1 \circ g_1 \circ \xi_1$ transforms parallelopiped $Q_{\ell j}^k$ into the parallelopiped $r_1 H_{\ell j}^k = R_{\ell j}^k$. Both $Q_{\ell j}^k$ and $R_{\ell j}^k$ are parallelopipeds with 2-rational vertices, and the mapping $r_1 \circ h_1 \circ g_1 \circ \xi_1$ is volume-preserving.

Step 9. Using Lemma 2.3 twice, we construct an isotopy s_t, $0 \le t \le 1$, of $R^k_{\ell j}$, such that $s_1 \circ r_1 \circ h_1 \circ g_1 \circ \xi_1$ is an affine mapping $\varphi^k_{\ell j} : Q^k_{\ell j} \to R^k_{\ell j}$.

Step 10. What remains is to construct an isotopy φ_t, connecting an affine volume preserving mapping φ of a parallelopiped Q into parallelopiped R (we omit indices j, k, ℓ) into some permutation σ of smaller equal cubes of some partition of Q and R. We shall describe the construction of this isotopy for the case $\nu = 2$; its generalization to higher ν is obvious, though more expansive.

Assume that the lengths of the sides of rectangle Q are $a \cdot 2^{-q}$ and $b \cdot 2^{-q}$, and for rectangle R are $c \cdot 2^{-q}$ and $d \cdot 2^{-q}$, $a, b, c, d \in \mathbb{Z}$, $a \cdot b = c \cdot d$. Partition Q and R into $a \cdot b$ equal squares κ with length of side 2^{-q}. Let $Q_{(q)} = \{\kappa \subset Q\}$, $R_{(q)} = \{\kappa \subset R\}$; let $\mathcal{F}_{(q)}(Q, R)$ be the set of all measure preserving, almost everywhere one-to-one mappings $f : Q \to R$, such that f is smooth in each $\kappa \in Q_{(q)}$, and continuous in $\overline{\kappa}$. Let $\mathcal{D}_{(q)}(Q, R) \subset \mathcal{F}_{(q)}(Q, R)$ be the set of mappings f such that, for each $\kappa \in Q_{(q)}$, $f\big|_\kappa$ is a shift of κ into some square $\kappa' \in R_{(q)}$. For a path $f_t \subset \mathcal{F}_{(q)}(Q, R)$ we define an action $J\{f_t\}^1_{t=0}$ in the usual way:

$$J\{f_t\}^1_{t=0} = \int_0^1 dt \int_Q \left| \frac{\partial \varphi_t(x)}{\partial t} \right|^2 dx \ .$$

Let $\varphi : Q \to R$ be an affine, area preserving mapping. We shall construct a flow $\varphi_t \subset \mathcal{F}_{(q)}(Q, R)$, $0 \le t \le 1$, such that $\varphi_0 = \varphi$, $\varphi_1 \in \mathcal{D}_{(q)}(Q, R)$, and $J\{\varphi_t\}^1_{t=0} < \infty$. To do this, partition R into rectangles R_i, containing one horizontal row of squares $\varphi(\kappa)$, $\kappa \in Q_{(q)}$. Let g_i be an affine mapping of R_i into the unit square K^2; let ρ be a mapping of K^2 into the unit circle B^2, described in Lemma 1.2; let ψ_α be a rotation of B^2 by the angle $(\pi \cdot \alpha)$. Then we describe the flow φ_t $(0 \le t \le \frac{1}{2})$ in the following way:

$$\varphi_t\big|_\kappa = g_i^{-1} \circ \rho^{-1} \circ \psi_t \circ \rho \circ g_i \circ \varphi\big|_\kappa \ ,$$

if $\varphi(\kappa) \subset R_i$.

The mapping $\varphi_{\frac{1}{2}}$ transforms all squares $\kappa \in Q_{(q)}$ into equal rectangles, and all these $a \cdot b$ rectangles are situated in R one above the other.

Now, partition R into c equal rectangles R^j, each of them containing d rectangles $\varphi_{\frac{1}{2}}(\kappa)$, $\kappa \in Q_{(q)}$. Let $h_j : R^j \to K^2$ be affine mappings. Define the flow $\varphi_t (\frac{1}{2} \le t \le 1)$,

$$\varphi_t\big|_\kappa = h_j^{-1} \circ \rho^{-1} \circ \psi_{\frac{1}{2}-t} \circ \rho \circ h_j \circ \xi_{\frac{1}{2}}\big|_\kappa \ ,$$

if $\xi_{\frac{1}{2}}(\kappa) \subset R^j$. It is evident that if $\kappa \in Q_{(q)}$, then $\varphi_1(\kappa) = \kappa' \in R_{(q)}$; so the desired flow φ_t is constructed.

The action of such an isotopy must not be small; to do it arbitrarily small, partition Q into m^ν equal parts, similar to Q, and reproduce this construction for each of them. Then the action of this isotopy (call it φ_t^m) will be m^2 times less than the action of φ_t. Taking m sufficiently large (and of the form the power of 2), we will reach arbitrarily small action. After performing the 10 construction steps, we reach a mapping $\varphi_1 \in \mathcal{F}_N$ (for some N, power of 2), that is a permutation of cubes, and the action of each of 10 steps of isotopy may be done arbitrarily small, if we choose appropriate parameters.

So , Lemma 2.2 is proved for the cube K.

Now, let ξ be a volume preserving diffeomorphism of an annulus domain $K' = K_{\varepsilon,\delta}$. Since $K_{\varepsilon,\delta} \subset K$, ξ may be regarded as a volume preserving diffeomorphism of K, the previous construction may be done. It can be seen immediately, that each of the 10 steps of this construction delivers mappings belonging to $\mathcal{F}_N(K')$, because all the mappings and isotopies involved are fixed in $K \backslash K'$. So we obtain the desired isotopy for the case $\mathcal{D}(K')$ as well.

3. Discrete Flows

In this section we prove the discrete version of Theorem 1.5. Our considerations apply, with obvious modifications, for all $\nu \geq 2$, but for $\nu > 2$ they are much more expansive. So we shall restrict ourselves to the case $\nu = 2$. Let $K'_N = \{\kappa \in K_N \mid \operatorname{dist}(\kappa, \partial K) < \delta\}$, $\mathcal{D}_N(K'_N) = \{\sigma \in \mathcal{D}_N \mid \sigma\kappa = \kappa, \ \text{ if } \kappa \neq K'_N\}$

THEOREM 3.1. *There is $C > 0$ such that for each N, and each $\sigma \in \mathcal{D}_N(K'_N)$ there exists a discrete flow $\sigma_1, \ldots, \sigma_k \in \mathcal{D}_N(K'_N)$, transforming σ into Id, such that its duration $k \leq C \cdot N$.*

We will refer to the following theorem from [S, Theorem 3.2] (it is proved there for all $\nu \geq 1$).

THEOREM 3.2. *Let $K_{M_1,\ldots,M_\nu} \subset K_N$ be a parallelopiped with lengths of edges $\frac{M_1}{N}, \ldots, \frac{M_\nu}{N}$, containing $M_1, \ldots, M_\nu$ cubes $\kappa \in K_N$; let $\mathcal{D}_N(K_{M_1,\ldots,M_\nu})$ be the group of such permutations that are fixed beyond $K_{M_1,\ldots,M_\nu}$. Then for each $\sigma \in \mathcal{D}_N(K_{M_1,\ldots,M_\nu})$ there is a discrete flow*

$\sigma_1, \dots, \sigma_k$ in $K_{M_1,\dots,M_\nu}$, transforming Id into σ, such that its duration $k \le C \cdot \max(M_1, \dots, M_\nu)$, where C depends only on ν.

A particular but especially useful case of this theorem is

LEMMA 3.3. *Let K_M be a row of M cubes κ and σ an arbitrary permutation of them. Then there exists a discrete flow $\sigma_1, \dots, \sigma_k$ in K_M transforming Id into σ and having duration $k \le C \cdot M$.*

Proof of Theorem 3.1: Partition $K' = \{x \in K \mid \operatorname{dist}(x, \partial K) < \delta\}$ into four rectangles (the square K is defined by inequalities $|x_i| < \frac{1}{2}$, $i = 1, 2$):

$$B_1 = \left\{(x_1, x_2) \mid \tfrac{1}{2}2 - \delta < x_1 < \tfrac{1}{2}, \quad -\tfrac{1}{2} < x_2 < \tfrac{1}{2}\right\} ;$$
$$B_2 = \left\{(x_1, x_2) \mid -\tfrac{1}{2} < x_1 < -\tfrac{1}{2} + \delta, \quad -\tfrac{1}{2} < x_2 < \tfrac{1}{2}\right\} ;$$
$$B_3 = \left\{(x_1, x_2) \mid -\tfrac{1}{2} + \delta < x_1 < \tfrac{1}{2} - \delta, \quad -\tfrac{1}{2} - \delta < x_2 < \tfrac{1}{2}\right\} ;$$
$$B_4 = \left\{(x_1, x_2) \mid -\tfrac{1}{2} + \delta < x_1 < \tfrac{1}{2} - \delta, \quad -\tfrac{1}{2} < x_2 < -\tfrac{1}{2} + \delta\right\} .$$

Let $B_{i,N} = \{\kappa \in K'_N \mid \kappa \in B_i\}$; $K'_N = \bigcup_{i=1}^{4} B_{i,N}$. For $k = 0, \dots, \delta \cdot N$ we define a closed row of squares (a "frame")

$$S_{k,N} = \left\{\kappa \in K'_N \mid \operatorname{dist}(\kappa, \partial K) = \frac{k}{N}\right\} .$$

The group $\mathcal{D}_N(K'_N)$ acts in a natural way on the functions $f(\kappa)$ on K'_N: for each $\tau \in \mathcal{D}_N(K'_N)$, $\tau f(\kappa) = f(\tau^{-1}\kappa)$. In particular, we define a function $c(\kappa)$, called *colour*: $c(\kappa) = i$, if $\kappa \in B_{N,i}$.

Let $\sigma \in \mathcal{D}_N(K'_N)$; we shall construct a discrete flow $\tau_1, \dots, \tau_p$ in K'_N, such that $\tau_p \circ \dots \circ \tau_1 \circ \sigma c(\kappa) = c(\kappa)$ for all $\kappa \in K'_N$, i.e. the flow $\tau_1, \dots, \tau_p$ returns each square κ to its original domain $B_{i,N}$. The duration p of this flow will satisfy the estimate $p \le C \cdot N$.

As a first step we shall construct a permutation φ_1, possessing the following properties:

(i) $\varphi B_{i,N} = B_{i,N}$ for all i,

(ii) φ_1 transforms the function $\sigma c(\kappa)$ into a function $\varphi_1 \circ \sigma c(\kappa) = c_1(\kappa)$, such that the number of squares κ, $c_1(\kappa) = 1$ (call them black squares), contained in $S_{k,N}$, is the number of the squares κ in $S_{k,N} \cap B_{i,N}$ for all k, $0 \le k \le \delta \cdot N$.

Permutation φ_1 is arbitrary up to the number of black squares $\kappa \in B_{i,N}$, such that $\varphi_1(\kappa) \in S_{k,N} \cap B_{i,N}$ $(i = 1, \ldots, 4)$. So we have to define this number for all i, k. It is sufficient to do this for $k = 1$, because after doing this, we obtain a problem of the same type in $K_N \setminus S_{1,N}$.

Let m_i be the number of black squares in $B_{i,N}$; $\mathcal{M}_i$ is the total number of squares κ in $B_{i,N}$; q_i is the number of squares in $S_{1,N} \cap B_{i,N}$. We have to choose the number p_i of black squares in $B_{i,N}$, transferred in $S_{1,N} \cap B_{i,N}$, so that $0 \leq p_i \leq m_i$, and $\sum\limits_i p_i = q_1$ (note that $\sum\limits_i m_i = \mathcal{M}_1$). Let us show that this is possible. The maximal possible value of p_i is $p_i^+ = \min(q_i, m_i)$; the minimal possible value is $p_i^- = \max\bigl(0, q_i - (\mu_i - m_i)\bigr)$, and we may take arbitrary p_i, such that $p_i^- \leq p_i \leq p_i^+$. Let us show, that $\sum\limits_i p_i^- \leq q_1 \leq \sum\limits_i p_i^+$. We shall use the fact, that $\mathcal{M}_1 \leq \mathcal{M}_2 \leq \ldots \leq \mathcal{M}_4$, $(\mathcal{M}_1 - q_1) \leq (\mathcal{M}_2 - q_2) \leq \ldots \leq (\mathcal{M}_4 - q_4)$, which is clear from the definition of B_i. Let

$$C_i = \mathcal{M}_i - q_i ;$$

then

$$\sum_{i=1}^{4} p_i^- = \sum_{i=1}^{4} \max\bigl(0, m_i - \bar{c}_i\bigr) = \sum_{i \in I'} (m_i - c_i) \ ,$$

where $I' = \{i \mid m_i - c_i > 0\}$. So, $\sum\limits_{i=1}^{4} p_i^- = 0$, if $I = \emptyset$, and

$$\sum_{i=1}^{4} p_i^- = \sum_{i \in I'} m_i - \sum_{i \in I'} c_i \leq \sum_{i=1}^{4} m_i - \sum_{i \in I'} c_i \leq$$
$$\leq \sum_{i=1}^{4} m_i - c_1 = \mathcal{M}_1 - c_1 = q_1 \ ;$$

$$\sum_{i=1}^{4} p_i^+ = \sum_{i=1}^{4} \min(m_i, q_i) \geq \begin{cases} \sum\limits_{i=1}^{4} m_i \ , & \text{if } m_i \leq q_i \text{ for all } i \\ q_j \ , & \text{if } m_j > q_j \text{ for some } j \ ; \end{cases}$$

in the first case $\sum\limits_{i=1}^{4} p_i^+ = \sum\limits_{i=1}^{4} m_i = \mathcal{M}_1 > q_1$; in the second case $\sum\limits_{i=1}^{4} p_i^+ \geq q_j \geq q_1$. So, we have proved, that $\sum\limits_{i=1}^{4} p_i^- \leq q_1 \leq \sum\limits_{i=1}^{4} p_i^+$, and we can choose p_i, $p_i^- \leq p_i \leq p_i^+$, such that $\sum\limits_{i=1}^{4} p_i = q_1$. Repeating this process for $k = 2, 3, \dots, \delta \cdot N$, we choose the number of the black squares in each $B_{i,N}$, transferred by the permutation φ_1 in $S_{k,N} \cap B_{i,N}$, so that the total number of such squares in $S_{k,N} \cap B_{i,N}$ will be $q_{1,k}$ = the number of squares in $S_{k,N} \cap B_{1,N}$.

Permutation φ_1 may be realized by some discrete flow $\tau_1, \dots, \tau_{p_1}$ of duration $p_1 \le C \cdot N$ (see Theorem 3.2).

Now, in each "frame" $S_{k,N}$ there is $q_{1,k}$ black squares κ, such that $\varphi_1 \circ \sigma c(\kappa) = 1$. Using Lemma 3.3, we organize the discrete flow $\tau_{p_1+1}, \dots, \tau_{p_2}$ (in each $S_{k,N}$, simultaneously and independently), transforming all the black squares into $S_{k,N} \cap B_{1,N}$. Let $\tau_{p_2} \circ \dots \circ \tau_{p_1+1} = \varphi_2$.

Now, let us define permutation φ_3, acting in $B_{2,N} \cup B_{3,N} \cup B_{4,N}$, preserving each $B_{i,N}$ and such that the number of squares κ of colour 2 in $B_{i,N} \cap S_{k,N}$, (i.e. such that $\varphi_3 \circ \varphi_2 \circ \varphi_1 \circ \sigma c(\kappa) = 2$), is $q_{2,k}$. This is done precisely as for φ_1. Let $\tau_{p_2+1}, \dots, \tau_{p_3}$ be discrete flow, realizing φ_3, according to Theorem 3.2 it may be chosen so that its duration $p_3 - p_2 \le C \cdot N$.

Now, we organize the discrete flow $\tau_{p_3+1}, \dots, \tau_{p_4}$ in each strip $(B_{2,N} \cup B_{3,N} \cup B_{4,N}) \cap S_{k,N}$ (simultaneously and independently), transforming all squares κ of colour 2 into $B_{2,N}$; its duration $p_4 - p_3$ may be chosen $\le C \cdot N$.

Finally, we construct the discrete flow $\tau_{p_4+1}, \dots, \tau_{p_6}$, transferring all the squares of colour 3 into $B_{3,N}$ (and automatically all the squares of colour 4 into $B_{4,N}$). Duration $p_6 - p_4 \le C \cdot N$. Now, we have constructed the discrete flow $\tau_1, \dots, \tau_{p_6}$, transforming σ into a permutation $\varphi \cdot \sigma$, such that $\varphi \circ \sigma B_{i,N} = B_{i,N}$ for all i. $B_{i,N}$ is a rectangle. Using Theorem 3.2, we may construct a discrete flow $\tau_{p_6+1}, \dots, \tau_{p_7}$ of duration $p_7 - p_6 < C \cdot N$, transforming each square κ to its original place: $\tau_{p_7} \circ \dots \circ \tau_1 \circ \sigma(\kappa) \equiv \kappa$ for all $\kappa \in K'_N$. The duration of the discrete flow $\tau_1, \dots, \tau_{p_7}$ is not more than $C \cdot N$. Thus Theorem 3.1 is proved for $\nu = 2$.

For $\nu \ge 3$ a similar construction may be used. We partition K'_N into parallelotopes $B_{i,N}$, and into shells $S_{k,N} = \{\kappa \mid \mathrm{dist}(\kappa, \partial K) = \frac{K}{N}\}$. Using alternating discrete flows in $B_{i,N}$ and in $S_{k,N}$, we return each cube κ to its original set $B_{i,N}$, and then use Theorem 3.2 to return it to its place. The duration of the discrete flow constructed is $\le C \cdot N$. Thus Theorem 3.1 is proved for all ν.

4. Proof of Lemma 1.5

Now we have everything to prove Lemma 1.5 and, therefore, Theorem A.

Let $K' = \{x \in K \mid \mathrm{dist}(x, \partial K)\} < \delta$, $K'_N = \{\kappa \in K_N \mid \kappa \cap K' \neq \emptyset\}$.

LEMMA 4.1. *Suppose that $\tau_1, \ldots, \tau_q$ is a discrete flow in K'_N, and its duration is q. Then there exists a discontinuous flow $\zeta_t \subset \mathcal{F}_N(K')$, $0 \le t \le 1$, such that $\zeta_{\frac{k}{q}} = \tau_k \circ \ldots \circ \tau_1 \in \mathcal{F}_N(K')$, $k = 0, \ldots, q$, and $J\{\zeta_t\}_{t=0}^1 \le CN^{-1}q\delta$ where C depends only on dimension.*

In other words, a discrete flow in K'_N may be realized by a discontinuous one, with the same estimate of action. This lemma is identical to Lemma 4.6 of [S], and the proof goes without modification.

Now we have a flow $\xi_t \subset \mathcal{D}(K')$, $0 \le t \le 1$, $\xi_0 = \xi$, $\xi_1 = Id$, such that $J\{\xi_t\}_{t=0}^1 < \infty$, but with no better estimate for the action; a flow $\eta_t \in \mathcal{F}_N(K')$, $0 \le t \le 1$, $\eta_0 = \xi$, $\eta_1 \in \mathcal{D}_N(K')$, and $J\{\eta_t\}_{t=0}^1$ may be done arbitrarily small, if N is chosen large enough; a flow $\zeta_t \subset \mathcal{F}_N(K')$, $0 \le t \le 1$, $\zeta_0 = \eta_1$, $\zeta_1 = Id$, and $J\{\zeta_t\}_{t=0}^1 < C \cdot \delta$. (This flow corresponds to the discrete flow $\tau_1, \ldots, \tau_q$, transforming $\sigma \in \mathcal{D}_N(K'_N)$ into Id, constructed in the preceding section.) Consider the composite path $\psi_t \subset \mathcal{F}_N(K')$, $0 \le t \le 1$,

$$\psi_t = \begin{cases} \xi_{3t}, & 0 \le t < \frac{1}{3} \\ \eta_{3t-1}, & \frac{1}{3} \le t < \frac{2}{3} \\ \zeta_{3t-2}, & \frac{2}{3} \le t \le 1 \,. \end{cases}$$

This path in $\mathcal{F}_N(K')$ is not yet precisely what we need. We require a small loop at the end of the path ψ_t. For each $\kappa \in K'_N$, let x_κ be the centre of the cube κ.

LEMMA 4.2. *Assume that $\nu \ge 3$. There exists a path $\chi_t \subset \mathcal{F}_N(K')$, $0 \le t \le 1$, such that $\chi_t(\kappa) = \kappa$ for all $\kappa \in K'_N$, $\chi_0 = \chi_1 = Id$ and such that for a composite path*

$$\varphi_t = \begin{cases} \psi_{2t}, & 0 \le t \le \frac{1}{2} \\ \chi_{2t-1}, & \frac{1}{2} \le t \le 1 \,, \end{cases}$$

the linear mapping $\varphi'_t(\chi_\kappa) \subset SL(\nu)$, $0 \le t \le 1$, is a contractible loop in $SL(\nu)$, for all $\kappa \in K'_N$. The action $J\{\chi_t\}_{t=0}^1 \le C \cdot \delta \cdot N^{-1}$.

Proof: It is well known that $\pi_1(SL(\nu))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $\nu \ge 3$. So we have to construct a smooth flow $|\chi_t|$ in each of $\kappa \in K'_N$, such that $\chi_0 = \chi_1 = Id$, and $\{\chi'_t(x_\kappa)\}_{t=0}^1$ is a unique nontrivial loop in $SL(\nu)$. This is done with the aid of Lemma 1.2. The estimate of action is evident.

Now everything is ready to prove Lemma 1.5. All the necessary properties are collected in

LEMMA 4.3. *Assume that* $\nu \geq 3$. *Let* $K' = \{x \in K \mid \mathrm{dist}(x, \partial K) < \delta\}$, $\xi \in \mathcal{D}(K')$, $\xi = Id$ *near* ∂K *and outside* K'; *let* $\varphi_t \subset \mathcal{F}_N(K')$, $0 \leq t \leq 1$, $\varphi_0 = \varphi_1 = Id$, $\varphi_t \subset \mathcal{D}(K')$ *for* $0 \leq t \leq \frac{1}{6}$, $\varphi_{\frac{1}{6}} = \xi$, $\{\varphi_t'(x_\kappa)\}_{t=0}^1$ is *contractible loop in* $SL(\nu)$ *for all* x_κ *– centres of the cubes* $\kappa \in K_N'$, *and*

$$J\{\varphi_t\}_{t=\frac{1}{6}}^1 < C \cdot \delta \, .$$

Then there exists a piecewise smooth path $\alpha_t \subset \mathcal{D}(K')$, $0 \leq t \leq 1$, $\alpha_0 = \xi$, $\alpha_1 = Id$, *and* $J\{\alpha_t\}_{t=0}^1 \leq C \cdot \delta$.

The path α_t coincides with a discontinuous flow $\varphi_{\frac{5}{6}t+\frac{1}{6}}$ everywhere except in a narrow neighbourhood of $\bigcup_{\kappa \in K_N'} \partial\kappa$, where the discontinuous flow is replaced by a smooth one. The procedure of smoothing is described in detail in [S, §5], and we shall not reproduce here the expansive constructions of this work. The only modification is that in addition to the "small" cubes $\kappa \in K_N'$ we have in our case a "large" cube $K \backslash K' = \{x \in K \mid \mathrm{dist}(x, \partial K) > \delta\}$. This is fixed under all the flows and transformations, and plays the same role as all the "small" cubes κ. Lemma 1.5 is just a tautological corollary of Lemma 4.3. Thus, Theorem A is proved.

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