Hawkes Process and Limit Order Book

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Hawkes Process

Definition (Hawkes Process 1971)

Consider N_t a counting process with the associated filtration $\mathcal{F}$ that satisfies the following conditions:

$$\mathbf{P}\left(N_{t+h} - N_t = 0 \,|\mathcal{F}_t\right) = 1 - \lambda_t h + o(h)$$

$$\mathbf{P}\left(N_{t+h} - N_t = 1 \,|\mathcal{F}_t\right) = \lambda_t h$$

$$\mathbf{P}\left(N_{t+h} - N_t > 1 \,|\mathcal{F}_t\right) = o(h)$$

where

$$\lambda_t = \mu + \int_0^{t_-} c e^{-a(t-s)} \mathrm{d}N_s$$

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where

$$\begin{aligned} \lambda_t &= \mu + \int_0^{t_-} c e^{-a(t-s)} \mathrm{d}N_s \\ &= \mu + \sum_{T_i < t} c e^{-a(t-T_i)} \end{aligned} \tag{1}$$

$\mu > 0$, $c > 0$ and, $a > 0$

Hawkes Process 2

The intensity process can be seen as a solution of the following SDE:

$$\mathrm{d}\lambda_t = a\left(\mu - \lambda_t\right)\mathrm{d}t + c\,\mathrm{d}N_t$$

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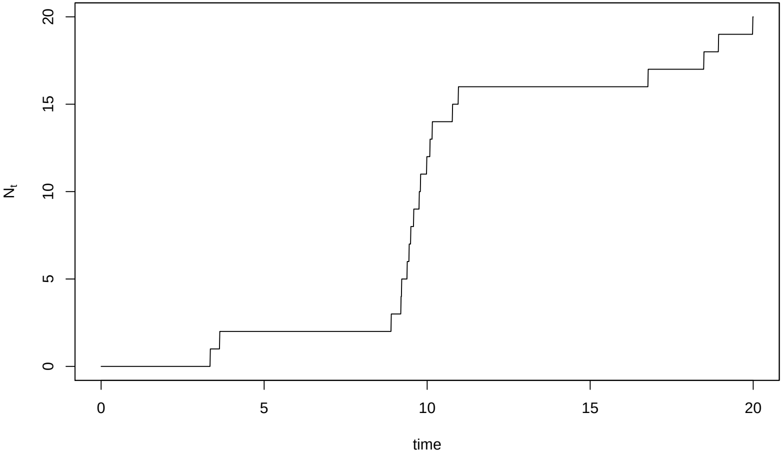
where the initial condition $\lambda_0 = \mu$. Indeed applying the stochastic calculus we get the general solution:

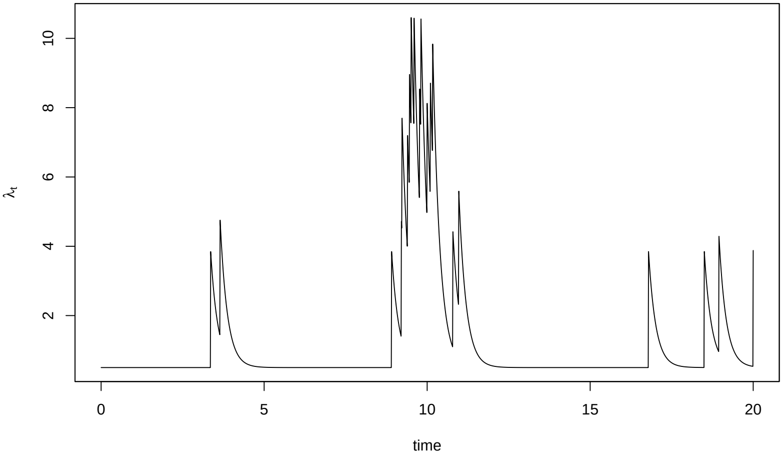
$$\lambda_t = e^{-at}\left(\lambda_0 - \mu\right) + \mu + \int_0^{t_-} ce^{-a(t-s)}\mathrm{d}N_s$$

Hawkes Process 3

Hawkes process with parameters $\mu = 0.5$, $c = 3.5$ $a = 4.5$



Hawkes Process 4



Hawkes Process 5

The hawkes process can be seen as a special case of the Point Process Regression Model where

$$g\left(t, Y_{t-}, \theta\right) = \mu$$

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Alternative Kernels:

- Simplified Power Law: $\frac{\alpha}{(t-s+1)^{\beta}}$
- Generalized Power Law: $\sum_{i=1}^{P} \frac{\alpha_i}{(t-s+\gamma_i)^{\beta_i}}$
- Generalized Exponential I: $\sum_{j=0}^{P} \alpha_j e^{-\beta_j(t-s)}$
- Generalized Exponential II: $\sum_{j=0}^{P} \alpha_j (t-s)^j e^{-\beta_j(t-s)}$
- Weibull: $\sum_{i=1}^{P} \alpha_i \frac{v_i(t-s)^{v_i-1}}{w_i^{v_i-2}} e^{-\left(\frac{t-s}{w_i}\right)^{v_i}}$
- Gamma: $\sum_{i=1}^{P} \frac{\alpha_i(t-s)^{d_i-1}}{\Gamma(d_i) b_i^{d_i-2}} e^{-\frac{t-s}{b_i}}$

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we need to evaluate $\mathbb{E}[\mathrm{d}N_s]$

Hawkes Process 7

Start with

$$\lambda_s = \lim_{h \to 0} \frac{\mathbb{E}\left[N_{s+h} - N_s \,|\mathcal{F}_s\right]}{h} = \frac{\mathbb{E}\left[\mathrm{d}N_s \,|\mathcal{F}_s\right]}{\mathrm{d}s}$$

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and take expectations

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to see that

$$\mathbb{E}\left[\mathrm{d}N_s\right] = \mathbb{E}\left[\lambda_s\right]\mathrm{d}s.$$

Hawkes Process 8

Therefore

$$\mathbb{E}\left[\lambda_t\right] = \mu + \int_0^{t_-} c e^{-a(t-s)} \mathbb{E}\left[\lambda_s\right] \mathrm{d}s$$

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$$\bar{\lambda} = \frac{\mu}{1-n}$$

The process does not explode when $n = \frac{c}{a} < 1$

Multidimensional Hawkes Process

We denote with $N_t = \left(N_t^1, \ldots, N_t^d\right)$ a d-dimensional counting process with the following intensities $\lambda_{t,i}$ $i = 1, \ldots, d$:

$$\lambda_{t,i} = \mu_i + \sum_{j=1}^{d} \int_0^t c_{i,j} e^{a_{i,j}(t-s)} \mathrm{d}N_t^j$$

What is a Limit Order Book?

It is essentially a file in a computer that contains all the orders sent to the market, with their characteristics such as the sign of the order (buy or sell), the price, the quantity, a timestamp giving the time the order was recorded by the market, and often, other market-dependent information.

What is a Limit Order Book?

The limit order book contains, at any given point in time, on a given market, the list of all the transactions that one could possibly perform on this market. Its evolution over time describes the way the market moves under the influence of its participants.

What is a Limit Order Book?

A market in which buyers and sellers meet via a limit order book, is called an **order-driven market**. In order-driven markets, buy and sell orders are matched as they arrive over time, subject to some priority rules. Priority is always based on price, and then, in most markets, on time, according to a FIFO (First in, first out) rule.

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- **Cancellation order**: an order to cancel an existing limit order.

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We define the *mid-price* as the average between the bid and the ask:

$$P_{mid} = \frac{P^A + P^B}{2}$$

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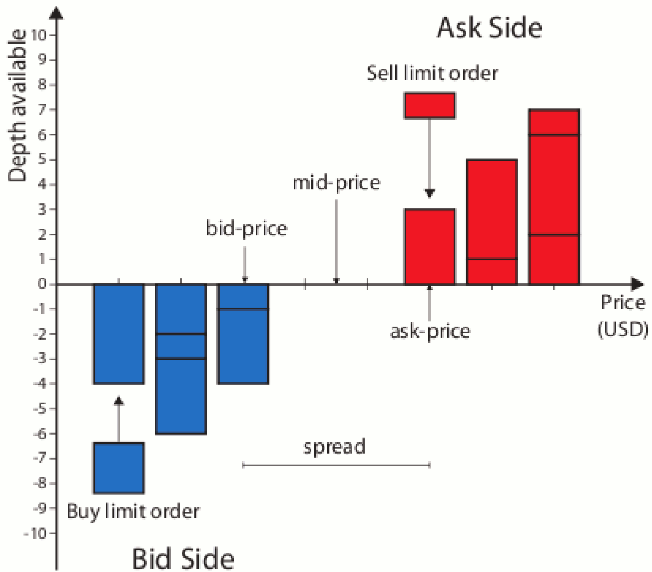

Figure 1: Limit Order Book

How to model the Limit Order Book

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  In this context we can use the multivariate Hawkes process to model the different type of orders.

A simple Multivariate Hawkes process for LOB

Inputs:

- N^L: Counting process of the Limit order
- N^M: Counting process of the Limit order

$$
\begin{aligned}
\lambda_t^M &= \mu^M + \int_0^t \alpha_{MM} e^{-\beta_{MM}(t-s)} \mathrm{d}N_s^M \\
\lambda_t^L &= \mu^L + \int_0^t \alpha_{LM} e^{-\beta_{LM}(t-s)} \mathrm{d}N_s^M + \int_0^t \alpha_{LL} e^{-\beta_{LL}(t-s)} \mathrm{d}N_s^L
\end{aligned}
$$

- MM and LL: self-exciting effect
- LM: When a market order is submitted, the intensity of the limit order process N^L increases.