Lecture 15
Asymptotic expansion methods

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Asymptotic Expansion

We will discuss:

- **Introduction to asymptotic expansion**
- **Small diffusion and asymptotic expansion**
- **Asymptotic expansion by YUIMA: pricing options**

Introduction to asymptotic expansion

What is asymptotic expansion?

- central limit theorem $Z_n \to^d N(0,1) \Leftrightarrow$

$$E[f(Z_n)] - \int f(z)\phi(z)dz \to 0 \quad (n \to \infty)$$

for $f \in C_b$, where ϕ is a normal density.

- asymptotic expansion (first order)

$$E[f(Z_n)] - \int f(z)\Big\{\phi(z) + n^{-1/2}p_1(z)\Big\}dz = o(n^{-1/2})$$

uniformly in a class of measurable functions f.

Why asymptotic expansion?

Asymptotic expansion is one of the fundamentals in

- **higher-order inferential theory**
- **prediction**
- **model selection, information criteria**
- **bootstrap and resampling methods**
- **information geometry**
- **stochastic numerical analysis**

Some references on the asymptotic expansion

- **independent sequences**
  - **Bhattacharya and Ranga Rao (1986)**
- **Markov chain**
  - **Götze and Hipp (ZW1983, AS1994)**
- **semimartngales**
  - **Mykland (AS1992) for differentiable *f***
  - **Y (PTRF1997, 2004)**
  - **Kusuoka and Y (PTRF2000)**
- **martingale expansion around a mixed normal limit Y (SPA2013, arXiv2012) (discussed in this talk)**
- **an exposition**
  - **Y (Chap 2 of "Rabi N. Bhattacharya", Springer 2016)**

Regularity of the distribution

- Bernoulli trials X_j $(n \in \mathbb{N})$ independent, $P[X_j = -1] = P[X_j = 1] = 1/2$
- F_n: the distribution function of $n^{-1/2} \sum_{j=1}^{n} X_j$
- For even $n \in \mathbb{N}$,

$$F_n(0) - F_n(0-) = P\left[\sum_{j=1}^{n} X_j = 0\right] = \binom{n}{n/2}\left(\frac{1}{2}\right)^n \sim C n^{-1/2}$$

- For <u>any</u> sequence of continuous functions Φ_n,

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \Phi_n(x) \right| \geq c\, n^{-1/2}$$

- Regularity of the distribution is essential.

Regularity of the distribution

- Asymptotic expansion is valid for

$$Z_n = n^{-1/2} \sum_{j=1}^{n} (X_j - E[X_j])$$

for i.i.d. sequence if $\mathcal{L}\{X_j\}$ has absolutely continuous part.

- Malliavin calculus ensures existence of density
- Natural to use the Malliavin calculus for functionals of semimartinagales

Small diffusion and asymptotic expansion

What we want to do

- pricing option
- Black-Scholes model

$$dX_t = 0.1X_t dt + \epsilon X_t dw_t, \quad X_0 = 100$$
$$\epsilon = 0.5, \quad K = 100$$

- Compute the price of Asian Call Option $(r = 0)$

$$C = E\left[\left(\frac{1}{T}\int_0^T X_t dt - K\right)^+\right]$$

- Monte Carlo method (time-consuming/iaccurate)
- asymptotic expansion (fast and fairly accurate)

$$C \sim c_0 + \epsilon c_1 + \epsilon^2 c_2 + \cdots + \epsilon^k c_k \quad (\epsilon \downarrow 0)$$

Small diffusion: a setting

- a diffusion process $X^\epsilon = (X_t^\epsilon)_{t\in[0,T]}$ satisfying

$$\begin{aligned} dX_t^\epsilon &= V_0(X_t^\epsilon, \epsilon)dt + V(X_t^\epsilon, \epsilon)dw_t, \quad t \in [0, T] \\ X_0^\epsilon &= x_0, \end{aligned} \tag{1}$$

- x_0 is a given vector,
- $V_0 \in C_\beta^\infty(\mathbb{R}^d \times [0,1], \mathbb{R}^d)$,
- $V = [V_1, ..., V_r] \in C_\beta^\infty(\mathbb{R}^d \times [0,1], \mathbb{R}^d \otimes \mathbb{R}^r)$,
- $w_t = [w_t^1, ... w_t^r]'$ is an r-dimensional standard Wiener process.

Functional of $X^\epsilon = (X_t^\epsilon)$

Many statistical problems of the perturbed system (1) entail a functional of the form

$$F^\epsilon = \sum_{\alpha=0}^{r} \int_0^T f_\alpha(X_t^\epsilon, \epsilon) dw_t^\alpha + F(X_T^\epsilon, \epsilon). \qquad (2)$$

Here

- $F, f_\alpha \in C_p^\infty(\mathbb{R}^d \times [0,1], \mathbb{R}^k)$ **(**$\alpha = 0, 1, ..., r$**) and** $w_t^0 = t$ **by convention.**
- **Our setting includes time-dependent stochastic differential equations and time-dependent functions** f_α **since we can always enlarge** X_t **to have the argument** t **if necessary.**

Example of the functional F^ϵ

- the price X_t^ϵ of a security
- the Black-Scholes economy:

$$\begin{aligned} dX_t^\epsilon &= cX_t^\epsilon dt + \epsilon X_t^\epsilon dw_t, \quad t \in [0,T], \\ X_0^\epsilon &= x_0. \end{aligned} \tag{3}$$

- A computational problem occurs when evaluating the average option as the expected value

$$E\left[\max\left\{\frac{1}{T}\int_0^T X_t^\epsilon dt - K, 0\right\}\right],$$

where K is a constant.

- The value of the option is given by composition of the smooth functional $F^\epsilon = \int_0^T X_t^\epsilon dt/T$ and the irregular function $\mathsf{f}(x) = \max\{x, 0\}$.

Go back to the general small diffusion model

- A diffusion process $X^\epsilon = (X_t^\epsilon)_{t\in[0,T]}$ and a functional F^ϵ:

$$\begin{cases} dX_t^\epsilon = V_0(X_t^\epsilon, \epsilon)dt + V(X_t^\epsilon, \epsilon)dw_t, \quad t \in [0, T] \\ X_0^\epsilon = x_0, \end{cases}$$

$$F^\epsilon = \sum_{\alpha=0}^{r} \int_0^T f_\alpha(X_t^\epsilon, \epsilon)dw_t^\alpha + F(X_T^\epsilon, \epsilon)$$

Deterministic limit condition

- Here we assume the deterministic limit condition

> [DL1] $V_\alpha(x,0) \equiv 0$ and $f_\alpha(x,0) \equiv 0$ for $\alpha = 1, ..., r$.

- Under Condition [DL1], the system X^0 in the limit is deterministic, described by the ordinary differential equation

$$
\begin{aligned}
dX_t^0 &= V_0(X_t^0, 0)dt, \quad t \in [0,T] \\
X_0^0 &= x_0.
\end{aligned} \tag{4}
$$

- A smooth eversion of X^ϵ exists: w.p.1, the mapping $(x, \epsilon) \mapsto X^\epsilon(t,x)$ is smooth.

Derivatives of X_t^ϵ

- $D_t^0 = \partial_\epsilon X_t^\epsilon|_{\epsilon=0}$:

$$\begin{aligned} dD_t^0 &= \partial_x V_0(X_t^0, 0)[D_t^0]dt + \partial_\epsilon V_0(X_t^0, 0)dt \\ &\quad + \partial_\epsilon V(X_t^0, 0)dw_t, \quad t \in [0, T] \\ D_0^0 &= 0. \end{aligned}$$

- D_t^0 is a Gaussian process

$$D_t^0 = Y_t \int_0^t Y_s^{-1} \left\{ \partial_\epsilon V_0(X_s^0, 0)ds + \sum_{\alpha=1}^r \partial_\epsilon V_\alpha(X_s^0, 0)dw_s^\alpha \right\}.$$

where $GL(d)$-valued process $Y = (Y_t)$ is a unique solution to the ODE

$$\frac{dY_t}{dt} = \partial_x V_0(X_t^0, 0)Y_t, \;\; Y_0 = I_d.$$

- The second derivative $E_t^0 = \partial_\epsilon^2 X_t^\epsilon|_{\epsilon=0}$ is a 2nd order polynomial of w_t.
- Similarly, the higher-order derivatives $\partial_\epsilon^i X_t^\epsilon$ are a solution of a higher-order SDE, and represented by multiple Wiener integrals

$$\int \cdots \int \int * * * * * dw_{t_1}^{\alpha_1} dw_{t_2}^{\alpha_2} \cdots dw_{t_n}^{\alpha_n}$$

- On the other hand, the derivatives of the functional

$$F^\epsilon = \sum_{\alpha=0}^{r} \int_0^T f_\alpha(X_t^\epsilon, \epsilon) dw_t^\alpha + F(X_T^\epsilon, \epsilon)$$

are represented by $\partial_\epsilon^i X_t^\epsilon$ $(i = 0, 1, ...)$.

Smooth stochastic expansion of F^ϵ

- **Expand F^ϵ around $\epsilon = 0$.**
- **We always take a smooth version of the solution X^ϵ: with probability 1, the mapping $(x, \epsilon) \mapsto X^\epsilon(t, x)$ is smooth.**
- **Apply the Taylor formula to obtain**

$$F^\epsilon = F^0 + \epsilon F^{[1]} + \epsilon^2 F^{[2]} + \cdots + \epsilon^{J-1} F^{[J-1]} + \epsilon^J R^{[J]}(\epsilon) \quad (5)$$

with

$$R^{[J]}(\epsilon) = \frac{1}{(J-1)!} \int_0^1 (1-s)^{J-1} \partial_\epsilon^J F^\epsilon|_{\epsilon = s\epsilon}\, ds.$$

- **Here $F^{[j]} = (j!)^{-1} \partial_\epsilon^j F^\epsilon|_{\epsilon=0}$ and $\partial_\epsilon^j F^\epsilon$ are expressed by $\partial_x^\nu \partial_\epsilon^n f_\alpha$, $\partial_x^\nu \partial_\epsilon^n F$ and $\partial_\epsilon^n X_t^\epsilon$.
Everything is a "polynomial of w_α".**

Smooth stochastic expansion of F^ϵ

- Since

$$F^0 = \int_0^T f_0(X_t^0, 0)dt + F(X_T^0, 0) \tag{6}$$

is a deterministic number, it is more natural to deal with the normalized functional

$$\tilde{F}^\epsilon = \epsilon^{-1}\left(F^\epsilon - F^0\right).$$

- Then $\tilde{F}^\epsilon$ admits an expansion corresponding to (5)

$$\tilde{F}^\epsilon = \tilde{F}^{[0]} + \epsilon\tilde{F}^{[1]} + \epsilon^2\tilde{F}^{[2]} + \cdots + \epsilon^{J-1}\tilde{F}^{[J-1]} + \epsilon^J\tilde{R}^{[J]}(\epsilon), \tag{7}$$

where $\tilde{F}^{[j]} = F^{[j+1]}$ and $\tilde{R}^{[J]}(\epsilon) = R^{[J+1]}(\epsilon)$.

$\tilde{F}^{\epsilon} = \tilde{F}^{[0]} + \epsilon \tilde{F}^{[1]} + \epsilon^2 \tilde{F}^{[2]} + \cdots$

- **In particular, under Condition [DL1],**

$$\tilde{F}^{[0]} = \int_0^T \partial_{(x,\epsilon)} f_0(X_t^0, 0)[(D_t^0, 1)]dt + \sum_{\alpha=1}^{r} \int_0^T \partial_\epsilon f_\alpha(X_t^0, 0) dw_t^\alpha + \partial_{(x,\epsilon)} F(X_T^0, 0)[(D_T^0, 1)], \quad (8)$$

$$\begin{aligned} \tilde{F}^{[1]} = & \frac{1}{2}\int_0^T \partial_x f_0(X_t^0, 0)[E_t^0]dt + \frac{1}{2}\int_0^T \partial^2_{(x,\epsilon)} f_0(X_t^0, 0)[(D_t^0, 1)^{\otimes 2}]dt \\ & + \frac{1}{2}\sum_{\alpha=1}^{r} \int_0^T \left\{ 2\partial_x \partial_\epsilon f_\alpha(X_t^0, 0)[D_t^0] + \partial_\epsilon^2 f_\alpha(X_t^0, 0) \right\} dw_t^\alpha \\ & + \frac{1}{2}\partial_x F(X_T^0, 0)[E_T^0] + \frac{1}{2}\partial^2_{(x,\epsilon)} F(X_T^0, 0)[(D_T^0, 1)^{\otimes 2}] \quad (9) \end{aligned}$$

where $D_t^0 = \partial_\epsilon X_t^0$ **and** $E_t^0 = \partial_\epsilon^2 X_t^0$**, and**

$\tilde{F}^{\epsilon} = \tilde{F}^{[0]} + \epsilon \tilde{F}^{[1]} + \epsilon^2 \tilde{F}^{[2]} + \cdots$

$$
\begin{aligned}
\tilde{F}^{[2]} = & \frac{1}{6}\int_0^T \partial_x f_0(X_t^0, 0)[C_t^0]dt + \frac{1}{2}\int_0^T \partial_x^2 f_0(X_t^0, 0)[D_t^0 \otimes E_t^0]dt \\
& + \frac{1}{2}\int_0^T \partial_x \partial_\epsilon f_0(X_t^0, 0)[E_t^0]dt + \frac{1}{6}\int_0^T \partial^3_{(x,\epsilon)} f_0(X_t^0, 0)[(D_t^0, 1)^{\otimes 3}]dt \\
& + \sum_{\alpha=1}^{r}\int_0^T \left\{ \frac{1}{2}\partial_x \partial_\epsilon f_\alpha(X_t^0, 0)[E_t^0] + \frac{1}{2}\partial_x^2 \partial_\epsilon f_\alpha(X_t^0, 0)[D_t^0 \otimes D_t^0] \right. \\
& \left. + \frac{1}{2}\partial_x \partial_\epsilon^2 f_\alpha(X_t^0, 0)[D_t^0] + \frac{1}{6}\partial_\epsilon^3 f_\alpha(X_t^0, 0) \right\} dw_t^\alpha \\
& + \frac{1}{2}\partial_x^2 F(X_T^0, 0)[E_T^0 \otimes D_T^0] + \frac{1}{2}\partial_x \partial_\epsilon F(X_T^0, 0)[E_T^0] \\
& + \frac{1}{6}\partial_x F(X_T^0, 0)[C_T^0] + \frac{1}{6}\partial^3_{(x,\epsilon)} F(X_T^0, 0)[(D_T^0, 1)^{\otimes 3}] \qquad (10)
\end{aligned}
$$

with $C_t^0 = \partial_\epsilon^3 X_t^0$.

Malliavin calculus, Watanabe theory

- smooth stochastic expansion

$$f^{\epsilon} \sim f^{[0]} + \epsilon f^{[1]} + \epsilon^2 f^{[2]} + \cdots \quad (\epsilon \downarrow 0)$$

(i.e., every residual term and its “derivatives” with respect to w are of small order.)

- g: function

$$g(f^{\epsilon}) \sim \sum_j \frac{1}{j!} \partial_x^j g(f^{[0]}) \big(\epsilon f^{[1]} + \epsilon^2 f^{[2]} + \cdots\big)^j$$
$$= \Phi^{[0]} + \epsilon \Phi^{[1]} + \epsilon^2 \Phi^{[2]} + \cdots .$$

In particular,

$$\Phi^{[0]} = g(f^{[0]})$$
$$\Phi^{[1]} = \partial_x g(f^{[0]}) f^{[1]}, \ldots$$

- **generalized expectation and asymptotic expansion**

$$E[g(f^{\epsilon})] \sim E[\Phi^{[0]}] + \epsilon E[\Phi^{[1]}] + \epsilon^2 E[\Phi^{[2]}] + \cdots .$$

- **Example.** $g(x) = \max\{x - K, 0\}$.

$$E[\partial_x^2 g(f^{[0]})\Psi] \ ?$$

- **Watanabe theory formulates**

$$E[T(f^{\epsilon})\psi], \ \ T: \text{ Schwartz distribution}$$

and drives asymptotic expansion.

- **formal computation**

$$\begin{aligned} E[\partial_x^2 g(f^{[0]})\Psi] &= E\Big[\partial_x^2 g(f^{[0]}) E[\Psi | f^{[0]}]\Big] \\ &= \int \partial_x^2 g(x) E[\Psi | f^{[0]} = x] \ p^{f^{[0]}}(x) dx \\ &= \int g(x) (-\partial_x)^2 \big(E[\Psi | f^{[0]} = x] \ p^{f^{[0]}}(x)\big) dx \end{aligned}$$

- **The (generalized) expectation of the j-th term in the expansion takes the form**

$$E[\Phi^{[j]}] = \int g(x) p_j(x) dx,$$

where $p_j(x)$ is expressed by the conditional expectation

$$E[(\text{a polynomial of } f^{[1]}, f^{[2]}, \ldots) | f^{[0]} = x]$$

Therefore, this problem is reduced to the computation of

$$E\Bigg[\int \cdots \int \int * * * dw_{t_1}^{\alpha_1} dw_{t_2}^{\alpha_2} \cdots dw_{t_n}^{\alpha_n} \,\Bigg|\, \int * * * dw_u = x \Bigg].$$

Smooth stochastic expansion of F^ϵ

Theorem 1. Suppose

- [DL1] and [DL2] (nondegeneracy condition)
- a sequence $(H^\epsilon)_{\epsilon\in(0,1]}$ of k'-dimensional random variables admits a smooth stochastic expansion

$$H^\epsilon \sim H^{[0]} + \epsilon H^{[1]} + \epsilon^2 H^{[2]} + \cdots \quad (\epsilon \downarrow 0). \tag{11}$$

- $\mathcal{G} \subset \mathcal{F}_\uparrow(\mathbb{R}^k)$

$$\sup_{g\in\mathcal{G}} \sup_x (1+|x|^2)^{-s/2} |g(x)| < \infty \tag{12}$$

  for some $s \geq 0$.

- Let $q \in C_\uparrow^\infty(\mathbb{R}^{k'})$.

Then ...

Then the expectation $E[g(\tilde{F}^\epsilon)q\,(H^\epsilon)]$ has an asymptotic expansion

$$E\left[g\left(\tilde{F}^\epsilon\right)q\,(H^\epsilon)\right] \sim c_0(g) + \epsilon c_1(g) + \epsilon^2 c_2(g) + \cdots \quad (\epsilon \downarrow 0),$$

where

$$c_i(g) = \int_{\mathbb{R}^k} g(z)p_i(z)dz \quad (g \in \mathcal{G})$$

for some densities $p_i(z)$. In particular,

$$p_0(z) = E\left[q\left(H^{[0]}\right)\middle|\tilde{F}^{[0]} = z\right]\phi(z;\mu,\Sigma)$$

$$p_1(z) = -\partial\cdot\left(E\left[q\left(H^{[0]}\right)\tilde{F}^{[1]}\middle|\tilde{F}^{[0]} = z\right]\phi(z;\mu,\Sigma)\right) \\ +E\left[\partial q\left(H^{[0]}\right)[H^{[1]}]\middle|\tilde{F}^{[0]} = z\right]\phi(z;\mu,\Sigma).$$

and

$$
\begin{aligned}
p_2(z) = \frac{1}{2}\partial^2 \cdot & \left(E\left[(\tilde{F}^{[1]})^{\otimes 2} q(H^{[0]}) \middle| \tilde{F}^{[0]} = z \right] \phi(z;\mu,\Sigma) \right) \\
& - \partial \cdot \left(E\left[\tilde{F}^{[2]} q(H^{[0]}) \middle| \tilde{F}^{[0]} = z \right] \phi(z;\mu,\Sigma) \right) \\
& - \partial \cdot \left(E\left[\tilde{F}^{[1]} \partial q(H^{[0]})[H^{[1]}] \middle| \tilde{F}^{[0]} = z \right] \phi(z;\mu,\Sigma) \right) \\
& + E\left[\partial q(H^{[0]})[H^{[2]}] \middle| \tilde{F}^{[0]} = z \right] \phi(z;\mu,\Sigma) \\
& + \frac{1}{2} E\left[\partial^2 q(H^{[0]})[(H^{[1]})^{\otimes 2}] \middle| \tilde{F}^{[0]} = z \right] \phi(z;\mu,\Sigma).
\end{aligned}
$$

This expansion is valid uniformly in $g \in \mathcal{G}$.

Remark 2. More than 900 terms are in the third order expansion. Each term is defined by a multiple integral.

Remark 3. Conditional expectation of a multiple Wiener integral given a Wiener integral is necessary and available.

Asymptotic expansion by YUIMA: pricing options

Asymptotic expansion vs. Monte Carlo method

- **YUIMA provides "asymptotic_term" function for asymptotic expansion of a functional of a diffusion process.**
- **At present, asymptotic_term is not available (out of order!) if the dimension of the diffusion process is greater than 1.**
- **A quite general, extended "expander" will be released soon.**

Asymptotic expansion vs. Monte Carlo method

- Compare the values obtained by asymptotic expansion and the Monte Carlo method, both by YUIMA
- Average Price Call Option for Black-Scholes model

$$dX_t = 0.1X_t dt + \epsilon X_t dw_t, \quad X_0 = 100$$
$$\epsilon = 0.5, \quad K = 100$$

- MC (10^5)
- Results:

method	*value*	*d.r.*
MC	14.56604	–
asy exp (1st order)	14.6774428	0.7648314%
asy exp (2st order)	14.5997711	0.2315927%

d.r.=difference rate

Asymptotic expansion vs. Monte Carlo method

- Average Price Call Option for CEV model

$$dX_t = 0.1X_t dt + \epsilon\sqrt{X_t}dw_t, \quad X_0 = 100 \ \ (\gamma = 1/2)$$
$$\epsilon = 0.5, \quad K = 100$$

- Results:

method	*value*	*d.r.*
MC	5.21069	–
asy exp (1st order)	5.2170308	0.1216788%
asy exp (2st order)	5.2170489	0.1220259%

d.r.=difference rate

Asymptotic expansion vs. Monte Carlo method

- **Average Price Call Option for CEV model**

$$dX_t = 0.1X_t dt + \epsilon\sqrt{X_t}dw_t, \quad X_0 = 100 \ \ (\gamma = 2/3)$$
$$\epsilon = 0.5, \quad K = 100$$

- **Results:**

method	*value*	*d.r.*
MC	5.915403	–
asy exp (1st order)	5.91404393	−0.02298126%
asy exp (2st order)	5.9134162	−0.0335933%

d.r.=difference rate

Asymptotic expansion vs. Monte Carlo method

- **The same method can be applied to Digital option, Basket option, Bermuda option.....**
- **VaR, CTE**
- **Control variable method**

Some references

- **Small σ expansion**
  - **Watanabe (AP1987), Kusuoka and Stroock (JFA1991)**
  - **Applications to statistics:**
    Y (PTRF1992,1993),
    Dermoune and Kutoyants (Stochastics1995),
    Sakamoto and Y (JMA1996, SISP1998),
    Uchida and Y (SISP2004),
    Masuda and Y (StatProbLet2004), ...

- **Application to option pricing:**
  Y (JJSS1992),
  Kunitomo and Takahashi (MathFinance2001),
  Uchida and Y (SISP2004),
  Takahashi and Y (SISP2004, JJSS2005),
  Osajima (SSRN2007),
  Takahashi and Takehara (2009,2010),
  Andersen and Hutchings (SSRN2009),
  Antonov and Misirpashaev (SSRN2009),
  Chenxu Li (ColumbiaUniv2010),
  ....

Comments

- **Today, we did not discuss really distributional asymptotic expansions.**
  - **Ex. Asy exp of the distribution of an ergodic process**
  - **Ex. Asy exp of the realized volatility**
  - **Ex. Asy exp of Skorohod integrals**
- **The theory of asymptotic expansion is now developing.**