Lévy process, Lévy driven SDE, and quasi-likelihood estimation *

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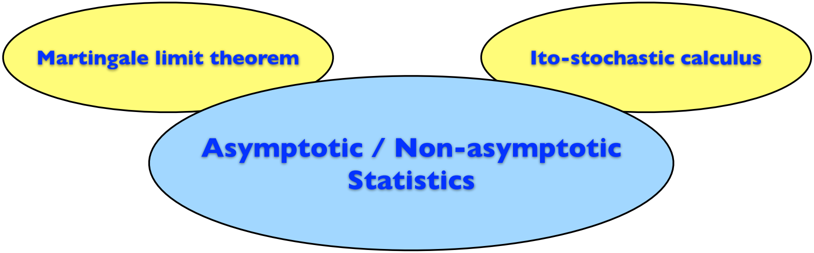
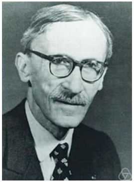
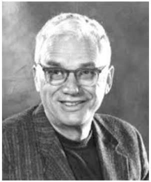


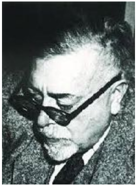


Paul Lévy
(1886-1971)

Kiyosi Itô
(1915-2008)



Joseph L. Doob
(1910-2004)



Norbert Wiener
(1894-1964)

- Standard references:
  - [Applebaum, 2009]
  - [Bertoin, 1996]
  - [Protter, 2005, (2nd.) Chapter I.4]
  - [Sato, 1999]
  - [Iacus and Yoshida, 2018] for many YUIMA examples

Discrete-time random walk $S_1, S_2, \ldots$

$$S_n := \sum_{j=1}^{n} \epsilon_j, \qquad S_0 := 0$$

- $\epsilon_1, \epsilon_2, \ldots$: i.i.d. random variables
- **Independent and stationary increments**

$$S_k - S_l = \sum_{j=l+1}^{k} \epsilon_j$$

1. $S_{j_1} - S_{j_0}, S_{j_2} - S_{j_1}, \ldots, S_{j_n} - S_{j_{n-1}}$ independent $(n \in \mathbb{N})$
2. $S_{j_k} - S_{j_{k-1}} \sim S_{j_k - j_{k-1}}$ $(k \in \mathbb{N})$

- Natural *continuous-time* counterpart?

Lévy process: Continuous-time random walk

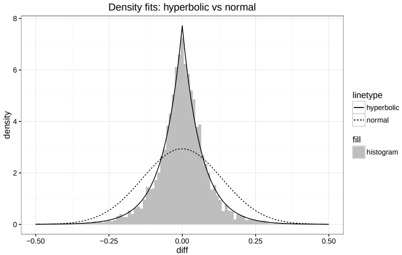
$$X_t = \sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}}) =: \sum_{j=1}^{n} \Delta_j X \quad (X_0 = 0 \text{ a.s.})$$

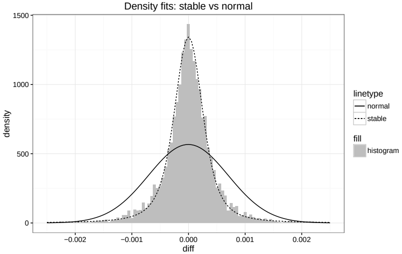
Definition

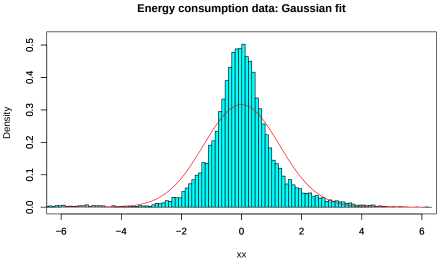
1. **Independent and stationary increments** $(0 = t_0 < t_1 < \cdots < t_n;\, n \in \mathbb{N})$
   - $X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.
   - $X_{t_j} - X_{t_{j-1}} \sim X_{t_j - t_{j-1}}$
2. Continuity in probability: $X_s \xrightarrow{p} X_t$ as $s \to t$.

- No pre-assigned jump time: $\mathbb{P}(|\Delta X_t| > 0) = 0$ for each $t > 0$.
- W.l.g. we may suppose that $t \mapsto X_t(\omega)$ is càdlàg.
- $\exists$Lévy process X s.t. $X_1 \sim F \iff F$ is **infinitely divisible**
  - F is infinitely divisible $\overset{\text{def.}}{\iff} \forall n \, \exists F_n,\ F = F_n^{*n} (:= F_n * \cdots * F_n)$

- Real data may be often leptokurtic: higher kurtosis than the normal (NYSE minutes data); maybe also skewed (energy consumption data).







Two prominent cases

- *If X is a counting process*:

$$X_t = \sum_{j \in \mathbb{N}} I(s \leq \tau_j) \text{ where } 0 < \tau_1 < \tau_2 < \dots \text{ are event-occurrence times,}$$

then X is necessarily a **Poisson process** with intensity λ:

$$\exists \lambda > 0, \quad X_t \sim \mathrm{Pois}(\lambda t), \quad t \in \mathbb{R}_+.$$

- *If X has continuous sample paths*, then X is necessarily a **Wiener process**:

$$\exists \mu \in \mathbb{R}\ \exists \sigma \geq 0, \quad X_t \sim N(\mu t, \sigma^2 t), \quad t \in \mathbb{R}_+,$$

i.e. we may write

$$X_t = \mu t + \sigma w_t$$

for a standard Wiener process w ($w_t \sim N(0, t)$).

Lévy-Khintchine representation[‡]

- Form of the Fourier transform:

$$\varphi_{X_t}(u) := \mathbb{E}(e^{iuX_t}) = \exp\{t\psi(u)\}, \quad u \in \mathbb{R},$$

with the characteristic exponent function

$$\psi(u) := iu\mu_1 - \frac{1}{2}\sigma^2 u^2 + \int \left(e^{iuz} - 1 - iuzI(|z| \leq 1)\right) \nu(dz).$$

- The element (μ_1, σ^2, ν) is called the **generating triplet** of Z:
  1. μ_1 is the **drift (location-shift)**,
  2. $\sigma^2 \geq 0$ is the **Gaussian variance**,
  3. ν is the **Lévy measure** (roughly, expected jump frequency).
     - $\int (1 \wedge |z|^2)\nu(dz) < \infty$ necessarily holds. [†]

[†] Can be of infinite mass: $\nu(\{|z| \leq 1\}) = +\infty$ in general, yet not too much.
[‡] Of dimension $d = 1$, while completely analogous for $d \geq 2$.

Remarks

- $\forall q > 0$: "$\mathbb{E}(|X_t|^q) < \infty \iff \int_{|z|>1} |z|^q \nu(dz) < \infty$"
- Mean and variance of X_t: if φ_{X_t} is of $\mathcal{C}^2$-class, then

$$\mathbb{E}(X_t) = i^{-1} t\psi'(0) = t\mu_1 + t\int_{|z|>1} z\nu(dz),$$
$$\mathrm{var}(X_t) = i^{-2} t\psi''(0) = t\sigma^2 + t\int z^2\nu(dz).$$

- kth cumulant of X_t: if φ_{X_t} is of $\mathcal{C}^k$-class ($k \geq 3$), then

$$i^{-k}\partial_u^k \log\varphi_{X_t}(0) = t\int z^k\nu(dz).$$

$$\frac{1}{t}\log\varphi_{X_t}(u) = iu\mu_1 - \frac{1}{2}\sigma^2 u^2 + \int \left(e^{iuz} - 1 - iuzI(|z|\le 1)\right)\nu(dz)$$

- In general, we should **not** do something like

$$\int \left(e^{iuz} - 1 - iuzI(|z|\le 1)\right)\nu(dz) \\ = \int \left(e^{iuz} - 1\right)\nu(dz) - iu\int_{|z|\le 1} z\nu(dz).$$

- The generating triplet uniquely determines the law of the **process** X, so that it determines e.g.

$$\mathcal{L}\Big(\sup_{t\in[0,1]}|X_t|\Big),\quad \mathcal{L}\big(\inf\{t\ge 0:\, |X_t|>1\}\big),\quad \mathcal{L}\Big(\int_0^1 X_s ds\Big).$$

Lévy-Itô decomposition of sample path

- Sum of independent Gaussian and non-Gaussian factors:

$$X_t = \mu_1 t + \sigma w_t + J_t$$

- More formally [Applebaum, 2009]:

$$X_t = t\mu_1 + \sigma w_t + \int_0^t \int_{|z|>1} z\mu(ds, dz) + \int_0^t \int_{|z|\le 1} z(\mu - \nu)(ds, dz). \qquad (1)$$

  - Poisson random measure $\mu((0,t], A) := \sum_{0<s\le t} I(\Delta X_s \in A \setminus \{0\})$
  - Semimartingale decomposition: local martingale plus finite-variation process.
  - The truncation level "1" could be any positive real instead.
- The jump-part form is mainly of theoretical importance; sometimes it may not be essential for simulation.

Toward generating discrete-time sample

Want to generate sample $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ from

$$X_t = \mu t + \sigma w_t + J_t,$$

where $0 = t_0 < t_1 < \cdots < t_n = t$ are (fine) sampling time points.

- Enough to be able to generate X_t for *any* $t > 0$:

$$X_t = \sum_{j=1}^{n} (X_{t_j} - X_{t_{j-1}}) = \sum_{j=1}^{n} \Delta_j X, \qquad \Delta_j X \sim X_{t_j - t_{j-1}}$$

- Simulator list in YUIMA [Brouste et al., 2014]:
  - Help documents of `rng` function in YUIMA.
  - [Iacus and Yoshida, 2018, Chapter 4]

Example: Compound Poisson process

$$X_t = \sum_{j=1}^{N_t} \epsilon_j$$

- $N_t \sim \mathrm{Pois}(\lambda t)$: Poisson process with intensity $\lambda > 0$.
- $\epsilon_1, \epsilon_2, \cdots \sim$ i.i.d. with $\mathbb{P}(\epsilon_1 = 0) = 0$, independent of N.
- *Any Lévy process can be a weak limit of a compound Poisson process.*
  - [Sato, 1999, Corollary 8.8]

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Example: Inverse-Gaussian subordinator

- The density of $X_t \sim \mathrm{IG}(\delta t, \gamma)$, $\delta, \gamma > 0$, is

$$x \mapsto \frac{\delta t e^{\delta t \gamma}}{\sqrt{2\pi}} x^{-3/2} \exp\left\{ -\frac{1}{2}\left(\frac{(\delta t)^2}{x} + \gamma^2 x \right) \right\}, \quad x > 0.$$

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Example: Normal inverse Gaussian Lévy process

- Normal variance-mean mixture, i.e. **subordination** §:

$$X_t = \mu t + \beta\tau_t + w_{\tau_t}$$

  - $\tau_t \sim \mathrm{IG}(\delta t, \gamma)$,
  - Standard Wiener process w independent of τ.

- The density of $X_t \sim NIG(\alpha, \beta, \delta t, \mu t)$ is

$$x \mapsto \frac{\alpha\delta t \exp\{\delta t\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu t)\}K_1(\alpha\psi(x;\delta t, \mu t))}{\pi\psi(x;\delta t, \mu t)}$$

  - $\alpha^2 := \gamma^2 + \beta^2$
  - $\psi(x;\delta t, \mu t) := \sqrt{(\delta t)^2 + (x - \mu t)^2}$

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§ General subordination formulae for probability and Lévy densities are available ($\tau \to X$); see [Iacus and Yoshida, 2018, Sect 4.8.3]; [Sato, 1999, chap 6] for general account.

Put simply

$$\varphi_{X_t}(u) := \mathbb{E}(e^{iuX_t}) = \exp\{t\psi(u)\}, \quad u \in \mathbb{R},$$
$$\psi(u) := iu\mu_1 - \frac{1}{2}\sigma^2 u^2 + \int \left(e^{iuz} - 1 - iuzI(|z| \le 1)\right) \nu(dz).$$

- Lévy process is completely characterized by the **generating triplet**, which
  - sometimes crucial in calculations,
  - while sometimes does not matter at all.
- Given any **infinitely divisible** distribution F, there essentially uniquely corresponds a Lévy process X such that $X_1 \sim F$.
- `rng` has several slots for generating specific Lévy process (see help file)
  - Approximate "inputting-$\nu(dz)$" way, yet to be implemented.

- A reader-friendly and comprehensive monograph is [Applebaum, 2009].

- Diffusion process is an SDE driven by a Wiener process:

$$dX_t = a(X_t)dt + b(X_t)dw_t,$$

a strong solution X realized as a functional form

$$X = F(X_0, w).$$

  - e.g. Geometric Brownian motion:

$$dX_t = X_t(\mu dt + \sigma dw_t),$$
$$X_t = X_0 \exp\left\{\sigma w_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right\}.$$

- The driving Wiener process w could be replaced by a **Lévy process**.

Lévy driven Stochastic Differential Equation (SDE)

$$dX_t = a(X_t)dt + b(X_t)dw_t + c(X_{t-})dJ_t$$

$$\left(X_t = x_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dw_s + \int_0^t c(X_{s-})dJ_s \right)$$

- Initial variable $X_0 \in \mathbb{R}^d$, possibly random.
- Driving noises:
  - d'-dimensional standard Wiener process $w = (w^j)_{j=1}^{d'}$;
  - d''-dimensional pure-jump Lévy process $J = (J^j)_{j=1}^{d''}$ of the form

$$J_t := \int_0^t \int_{|z|\le 1} z(\mu - \nu)(ds, dz) + \int_0^t \int_{|z|>1} z\mu(ds, dz).$$

- Coefficient functions:
  - Drift coefficient $a(x) = \{a_k(x)\}_{k\le d} : \mathbb{R}^d \to \mathbb{R}^d$
  - Diffusion coefficient $b(x) = \{b_{kl}(x)\}_{k\le d;\, l\le d'} : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^{d'}$
  - Jump coefficient $c(x) = \{c_{kl}(x)\}_{k\le d;\, l\le d''} : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^{d''}$

$$dX_t = a(X_t)dt + b(X_t)dw_t + c(X_{t-})dJ_t$$
$$\left(X_t = x_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dw_s + \int_0^t c(X_{s-})dJ_s \right)$$

- The stochastic integrations:

$$\int_0^t Y_{s-}dJ_s := \lim_{n\to\infty} \sum_{j=1}^n Y_{(j-1)t/n} \left(J_{jt/n} - J_{(j-1)t/n} \right)$$
$$= \int_0^t \int_{|z|>1} Y_s z\mu(ds, dz) + \int_0^t \int_{|z|\le 1} Y_s z(\mu - \nu)(ds, dz)$$

with the notation in (1), [Applebaum, 2009, Section 6.2];

- L^2-stochastic integration theory for small-jump part,
- Pathwise interlacing of large-jump component for large-jump part.

$$dX_t = a(X_t)dt + b(X_t)dw_t + c(X_{t-})dJ_t$$
$$\left(X_t = x_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dw_s + \int_0^t c(X_{s-})dJ_s \right)$$

- Globally Lipschitz (a, b, c): $\exists K > 0,\ \forall x_1, x_2 \in \mathbb{R}^d$,

$$|a(x_1) - a(x_2)| + |b(x_1) - b(x_2)| + |c(x_1) - c(x_2)| \le K|x_1 - x_2|,$$

  leads to existence of unique strong solution (a (w, J)-Lévy functional)

$$X =: F(x_0, w, J).$$

  - [Applebaum, 2009, Theorems 6.2.9 and 6.4.6].

- The simplest but widely applicable device to approximate a solution process to is the **Euler(-Maruyama) scheme** [Platen and Bruti-Liberati, 2010].

Euler-discretization scheme

- As in the case of diffusions, for $t_j - t_{j-1}$ small enough,

$$\begin{aligned} X_{t_{j+1}} &= X_{t_j} + \int_{t_j}^{t_{j+1}} a(X_s)ds + \int_{t_j}^{t_{j+1}} b(X_s)dw_s + \int_{t_j}^{t_{j+1}} c(X_{s-})dJ_s \\ &\approx X_{t_j} + \int_{t_j}^{t_{j+1}} a(X_{t_{j-1}})ds + \int_{t_j}^{t_{j+1}} b(X_{t_{j-1}})dw_s + \int_{t_j}^{t_{j+1}} c(X_{t_{j-1}})dJ_s \\ &\approx X_{t_{j-1}} + a(X_{t_{j-1}})(t_j - t_{j-1}) + b(X_{t_{j-1}})(w_{t_j} - w_{t_{j-1}}) \\ &\quad + c(X_{t_{j-1}})(J_{t_j} - J_{t_{j-1}}) \end{aligned} \tag{2}$$

Need to generate $J_{t_j} - J_{t_{j-1}} (\sim \Delta_j J)$-random number at each step.

- For this, YUIMA internally use the **rng** slots.

Generation of discretized process

The Euler-discretized process $X^\Delta =: (X_t^\Delta)$, for $\Delta > 0$ small enough:

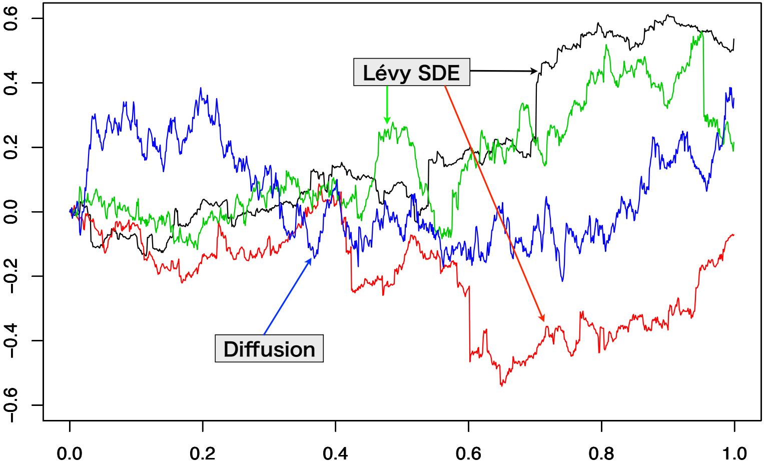
1. $X_t^\Delta := X_0$ for $t \in [0, \Delta)$.
2. For $t \in [j\Delta, (j+1)\Delta)$, $j \in \mathbb{N}$,

$$X_t^\Delta := X_{(j-1)\Delta}^\Delta + a_{j-1}\Delta + b_{j-1}\Delta_j w + c_{j-1}\Delta_j J.$$

- $f_{j-1} := f(X_{(j-1)\Delta})$
- $\Delta_j x = \Delta_j^n x := x_{t_j} - x_{t_{j-1}}$: the jth increment of a process x

- Then, we approximate as $X_t \approx X_t^\Delta$ over a period $[0, T]$;
  - Having generated a finest-approximating process X^Δ,
  - we can extract any thinned process, say $X^{k\Delta}$ for some $k \geq 2$,
  - which plays a role of discretely observed sample from X.
- Strong and weak approximation errors are defined as in diffusion cases.

- May seem like:



Example: Diffusion with compound-Poisson jumps

- For J begin a compound Poisson process with $\Gamma(3,3)$-distributed jumps,

$$dX_t = \{\sin(X_t) - X_t\}\, dt + 2dw_t - dJ_t.$$

  - Downward jumps only.

- The recipe YUIMA uses for $(X_t)_{t\in[0,T]}$:

$$dX_t = a(X_t)dt + b(X_t)dw_t + c(X_{t-})dJ_t$$

0. Generate X_0 and $N_T \leftarrow \mathrm{Pois}(\lambda T)$ independently, and set $j = 1$.
    - 0.1. If $N_T = 0$, then $(X_t)_{t\in[0,T]}$ is a diffusion $dX_t = a(X_t)dt + b(Z_t)dw_t$.
    - 0.2. Otherwise, generate $U_1, \dots, U_{N_T} \sim$ i.i.d.$U(0,T)$;
      Sort them as $U_{(1)} < U_{(2)} < \cdots < U_{(N_T)}$;
      For $k \le N_T$, pick a $j_k \in \{1, 2, \dots, [T/\Delta]\}$ s.t. $U_{(k)} \in ((j_k - 1)\Delta, j_k\Delta]$. ¶
1. Generate $\eta_j \sim N_{d'}(0, I_{d'})$ and then
    - 1.1. If $j = j_k$ for some k, then generate $\zeta_k \sim F(dz)$ (jump law) and deliver

$$X_{j\Delta} \leftarrow X_{(j-1)\Delta} + a_{j-1}\Delta + b_{j-1}\sqrt{\Delta}\,\eta_j + c_{j-1}\zeta_k.$$

    - 1.2. Otherwise, $X_{j\Delta} = X_{(j-1)\Delta} + a_{j-1}\Delta + b_{j-1}\sqrt{\Delta}\,\eta_j$.
2. Update $j = j + 1$ and return to step 1: repeat step 1 until $j = [T/\Delta]$.

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¶Ignores the possibility of multiple i_j: that'll be negligoble for Δ small enough.

Example: Geometric Lévy process

- SDE driven by a general Lévy process $X_t = \mu t + \sigma w_t + J_t$:

$$dY_t = Y_{t-} dX_t, \qquad Y_0 = 1,$$

$$Y_t = \exp\left(X_t - \frac{\sigma^2}{2} t\right) \prod_{0<s\le t} (1 + \Delta X_s) e^{-\Delta X_s}$$

  - Can be checked by **Itô's formula** (in the next page).
  - If in particular $J \equiv 0$, then Y is the geometric Brownian motion:

$$Y_t = \exp\left\{\sigma w_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right\}$$

- Yet, we apply Euler-Maruyama scheme directly.

Itô's formula (Univariate case)

$$dX_t = a(X_t)dt + b(X_t)dw_t + c(X_{t-})dJ_t$$

$$\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d\langle X^c\rangle_s \\
&\quad + \sum_{0<s\le t}\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\} \\
&= f(X_0) + \int_0^t \left(a(X_s)f'(X_{s-}) + \frac{1}{2}b^2(X_s)f''(X_{s-})\right) ds \\
&\quad + \int_0^t f'(X_{s-})b(X_s)dw_s + \int_0^t f'(X_{s-})c(X_{s-})dJ_s \\
&\quad + \sum_{0<s\le t}\{f(X_{s-} + c(X_{s-})\Delta J_s) - f(X_{s-}) - f'(X_{s-})c(X_{s-})\Delta J_s\}.
\end{aligned}$$

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Example: Heavy-tailed SDE

- Non-Gaussian infinite-variance Lévy process $J_t \sim S_\alpha(\beta, \sigma, \mu)$:

$$\varphi_{J_t}(u) = \begin{cases} -(t^{1/\alpha}\sigma)^\alpha |u|^\alpha \left(1 - i\beta \mathrm{sign}(u) \tan\frac{\alpha\pi}{2}\right) + i\mu t u, & \alpha \neq 1 \\ -t\sigma|u|\left(1 + i\frac{2\beta}{\pi}\mathrm{sign}(u)\log|u|\right) + i\mu t u, & \alpha = 1 \end{cases}$$

  - $(\alpha, \beta, \sigma, \mu) \in (0,2) \times [-1,1] \times (0,\infty) \times \mathbb{R}$:
    - $\alpha > 1 \Rightarrow$ Finite mean and infinite variance
    - $\alpha = 1 \Rightarrow$ Cauchy (possibly skewed)
    - $\alpha < 1 \Rightarrow$ Infinite mean
  - The index $\alpha \in (0,2)$ controls tail heaviness and small-jump activity.

- SDE driven by a Lévy process $J_1 \sim \mathrm{Stable}(1.3, 0, 1, 0)$:

$$dX_t = -\frac{X_t}{\sqrt{1 + X_t^2}} dt + dJ_t$$

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Example: Multidimensional nonlinear SDE

- The same recipe as in the one-dimensional case (2):

$$
\begin{aligned}
\underset{d\times 1}{X_{t_{j+1}}} &= X_{t_j} + \int_{t_j}^{t_{j+1}} \underset{d\times 1}{a(X_s)}ds + \int_{t_j}^{t_{j+1}} \underset{d\times r}{b(X_s)}\underset{r\times 1}{dw_s} + \int_{t_j}^{t_{j+1}} \underset{d\times m}{c(X_{s-})}\underset{m\times 1}{dJ_s} \\
&\approx X_{t_{j-1}} + a(X_{t_{j-1}})(t_j - t_{j-1}) + b(X_{t_{j-1}})(w_{t_j} - w_{t_{j-1}}) \\
&\qquad + c(X_{t_{j-1}})(J_{t_j} - J_{t_{j-1}})
\end{aligned}
$$

  - Just matrix multiplications, keeping the form:

(Predictable coefficient) $\times$ **(Noise increment)**

- YUIMA has slots for exact generation of multidimensional $J_{t_j} - J_{t_{j-1}}$.

- Two-dim. SDE $X = (X^1, X^2)$ driven by a 2-dim. NIG Lévy process:

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} -2X_t^1 \\ 0.3X_t^1 - 1/\sqrt{1+(X_t^2)^2} \end{pmatrix} dt + \begin{pmatrix} 1/\sqrt{1+(X_t^1)^2} & -0.5 \\ 0 & 1 \end{pmatrix} dJ_t$$

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Put simply

$$dX_t = a(X_t)dt + b(X_t)dw_t + c(X_{t-})dJ_t$$

$$J_t = \int_0^t \int_{|z|\le 1} z(\mu - \nu)(ds, dz) + \int_0^t \int_{|z|>1} z\mu(ds, dz)$$

- Good (a, b, c) leads to the existence of unique strong solution.
- `simulate` in YUIMA can generate X, as soon as YUIMA can generate J_h.
- YUIMA has several options for $\mathcal{L}(J_h)$-random numbers.

- Discrete-time location-scale time series model:

$$X_n = b(X_{n-1}, \beta) + a(X_{n-1}, \alpha)\epsilon_n$$

  - $\epsilon_1, \epsilon_2, \cdots \sim$ i.i.d. $(0, 1)$
  - $\theta = (\alpha, \beta)$: Statistical parameter, to be estimated from $(X_1, \dots, X_n)$.

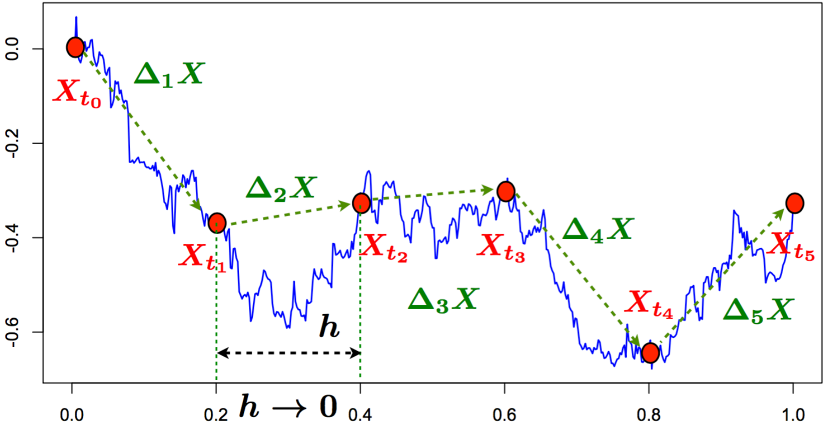
- A natural continuous-time counterpart is [a]

$$dX_t = a(X_{t-}, \alpha)dZ_t + b(X_t, \beta)dt, \qquad \theta = (\alpha, \beta)$$

  - Z is a **standard Lévy process**: $\mathbb{E}(Z_t) = 0$ and $\mathrm{var}(Z_t) = t$.
  - Estimate θ from $(X_{t_0}, X_{t_1}, \dots, X_{t_n})$.

[a] Note: the coefficient notation got changed! $(a \leftrightarrow b)$

- *High-frequency sampling* can provide us with *unified* inference strategies, which generally cannot be shared with the discrete-time framework.



Setup: joint asymptotics

- Univariate parametric Stochastic differential equation (SDE):

$$dX_t = a(X_{t-}, \alpha)dZ_t + b(X_t, \beta)dt, \qquad \theta = (\alpha, \beta)$$

- Available data $(X_{t_j})_{j=0}^n$; $t_j = jh_n = jh$, $T_n := nh \to \infty$, $nh^2 \to 0$.
- Driving Lévy process s.t. $\mathbb{E}(Z_t) = 0$, $\mathrm{var}(Z_t) = t$:

$$Z_t = \sigma W_t + \int_0^t \int z\,(\mu - \nu)(ds, dz),$$

  - A nuisance element.

Regularity conditions

$$dX_t = a(X_{t-}, \alpha)dZ_t + b(X_t, \beta)dt$$

- Correctly specified [‖] smooth coefficients, known up to $\theta := (\alpha, \beta)$.
- Stability ((Exponential) Ergodicity and bounded moments) [**]:

$$\frac{1}{T}\int_0^T f(X_s)ds \xrightarrow{p} \int f(x)\pi(dx), \quad T \to \infty, \tag{3}$$

$$\forall q > 0, \quad \sup_{t \in \mathbb{R}_+} \mathbb{E}\left(|X_t|^q\right) < \infty. \tag{4}$$

- Identifiability:

$$\text{“}(a(\cdot, \alpha), b(\cdot, \beta)) = (a(\cdot, \alpha_0), b(\cdot, \beta_0)),\ \pi\text{-a.e.”} \Rightarrow \theta = \theta_0$$

[‖] i.e. True data-generating parameter $\theta_0 = (\alpha_0, \beta_0)$ is assumed to exist.

[**] A reader-friendly and comprehensive monograph is [Kulik, 2018].

Remark on the stability assumption

$$dX_t = a(X_t)dt + b(X_t)dw_t + c(X_{t-})dJ_t.$$

- X is "exponentially" ergodic (hence (3)) and (4) if:
  1. (a, b, c) is of class $\mathcal{C}^1(\mathbb{R})$ and globally Lipschitz, and (b, c) is bounded.
  2. Either one of the following conditions holds true:
     (i) $b(x') \neq 0$ for some x', $c(x'') \neq 0$ for every x'', and there exists a constant $\overline{\epsilon} > 0$ such that $\nu(-\epsilon, 0) \wedge \nu(0, \epsilon) > 0$ for every $\epsilon \in (0, \overline{\epsilon})$;
     (ii) $b \equiv 0$, $c(x'') \neq 0$ for every x'', and we have the decomposition
     $$\nu = \nu_\star + \nu_\natural$$
     for two Lévy measures $\nu_\star$ and $\nu_\natural$, where the restriction of $\nu_\star$ to some open set of the form $(-\overline{\epsilon}, 0) \cup (0, \overline{\epsilon})$ admits a continuously differentiable positive density $g_\star$.
  3. $\mathbb{E}(J_1) = 0$ and $\int_{|z|>1} |z|^q \nu(dz) < \infty$ for every $q > 0$, and
     $$\limsup_{|x|\to\infty} \frac{a(x)}{x} < 0.$$
- See [Masuda, 2013, Sect 5] for details; another conditions are possible.

GQLF and GQMLE

$$dX_t = a(X_{t-}, \alpha)dZ_t + b(X_t, \beta)dt, \qquad \theta = (\alpha, \beta)$$

- Gaussian approximation in small time ($f_{j-1}(\theta) := f(X_{t_{j-1}}, \theta)$):

$$X_{t_j} \overset{\mathbb{P}_\theta}{\approx} X_{t_{j-1}} + a_{j-1}(\alpha)\Delta_j Z + hb_{j-1}(\beta)$$
$$\overset{\mathcal{L}(\mathbb{P}_\theta)}{\approx} X_{t_{j-1}} + a_{j-1}(\alpha)N(0,h) + hb_{j-1}(\beta)$$

Gaussian quasi-likelihood function (GQLF) and Gaussian QMLE (GQMLE)

$$\mathbb{H}_n(\theta) = \sum_{j=1}^{n} \log\phi\left(X_{t_j};\, X_{t_{j-1}} + hb_{j-1}(\beta),\, ha^2_{j-1}(\alpha)\right), \qquad (5)$$

$$\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n) \in \operatorname{argmax} \mathbb{H}_n.$$

- User's input:
    - Function forms of scale $a(x, \alpha)$ and drift $b(x, \beta)$.
    - Small sampling stepsize value $h = h_n$.

Asymptotic normality: joint asymptotics

$$dX_t = a(X_{t-},\alpha)dZ_t + b(X_t,\beta)dt, \qquad \theta = (\alpha,\beta)$$

- Diffusion case $Z = w$ [Kessler, 1997]:

$$\left(\sqrt{n}(\hat{\alpha}_n - \alpha_0),\ \sqrt{T_n}(\hat{\beta}_n - \beta_0)\right) \xrightarrow{\mathcal{L}} N_{p_\alpha+p_\beta}\left(0,\ \mathrm{diag}\{\mathcal{I}_\alpha^{-1}(\alpha_0), \mathcal{I}_\beta^{-1}(\theta_0)\}\right)$$

- In the presence of jumps [Masuda, 2013]

$$\left(\sqrt{T_n}(\hat{\alpha}_n - \alpha_0),\ \sqrt{T_n}(\hat{\beta}_n - \beta_0)\right) \xrightarrow{\mathcal{L}} N_{p_\alpha+p_\beta}\left(0,\ \begin{pmatrix} \nu_4 \mathcal{I}_\alpha^{-1}(\theta_0) & \text{sym.} \\ \nu_3 \mathcal{J}_{\alpha\beta}(\theta_0) & \mathcal{I}_\beta^{-1}(\beta_0) \end{pmatrix}\right)$$

- Straightforward to estimate $\mathcal{I}_\alpha(\theta_0)$, $\mathcal{I}_\beta(\beta_0)$, and $\mathcal{J}_{\alpha\beta}(\theta_0)$ empirically.
- Non-Gaussian structure of Z appears in the asymptotic covariance:

$$\nu_k := \int z^k \nu(dz), \qquad k = 3, 4.$$

- Difference in magnitude in small time:

$$X_{t_{j+1}} = X_{t_j} + \underbrace{\int_{t_j}^{t_{j+1}} b(X_s, \beta)ds}_{\approx O_p(h)} + \underbrace{\int_{t_j}^{t_{j+1}} a(X_{s-}, \alpha)dZ_s}_{\approx O_p(\sqrt{h})}$$

- Suggests:
  - First estimate α with ignoring $b(x, \beta)$;
  - Then estimate β with plugging in $\hat{\alpha}_n$,

  even for general standard Lévy process Z;
  - See [Kamatani and Uchida, 2015] and the ref's therein for the diffusion case.

Setup: stepwise asymptotics

Objective

- Estimate true parameter $\theta_0 = (\alpha_0, \gamma_0)$ of

$$dX_t = a(X_t, \alpha, \gamma)dt + c(X_{t-}, \gamma)dJ_t.$$

from discrete-time sample $(X_{t_j})_{j=1}^n$ for $t_j = jh_n$ with $h = h_n$ s.t.

  - $\exists \epsilon_0 \in (0, 1),\ nh^{1+\epsilon_0} \to \infty$
  - $nh^2 \to 0$

Assumed

- Smooth parametric coefficients known up to $\theta = (\alpha, \gamma) \in \mathbb{R}^p$.
- Standardized pure-jump Lévy process J s.t. $\forall q > 0,\ \mathbb{E}(|J_t|^q) < \infty$.
- Exponentially ergodicity and bounded moments.
- Identifiability

$$dX_t = a(X_t, \alpha, \gamma)dt + c(X_{t-}, \gamma)dZ_t$$

$$X_{t_j} \overset{\mathbb{P}_\theta}{\approx} X_{t_{j-1}} + ha_{j-1}(\alpha, \gamma) + c_{j-1}(\gamma)\Delta_j Z$$

Stepwise-estimation recipe [Uehara and Masuda, 2017], with $\mathbb{H}_n(\alpha, \gamma)$ of (5)

1. $\mathcal{L}(X_{t_j} | X_{t_{j-1}} = x) \overset{\mathbb{P}_\theta}{\approx} N\left(x,\ hc^2(x, \gamma)\right)$: $\quad \hat{\gamma}_n \in \operatorname{argmin}_\gamma \mathbb{H}_n^1(\gamma)$,

$$\mathbb{H}_n^1(\gamma) := \sum_{j=1}^{n} \log \phi\left(X_{t_j};\ X_{t_{j-1}},\ hc_{j-1}^2(\gamma)\right)$$

2. $\mathcal{L}(X_{t_j} | X_{t_{j-1}} = x) \overset{\mathbb{P}_\theta}{\approx} N\left(x + ha(x, \alpha, \gamma),\ hc^2(x, \hat{\gamma}_n)\right)$: $\hat{\alpha}_n \in \operatorname{argmin}_\alpha \mathbb{H}_n(\alpha, \hat{\gamma}_n)$,

$$\mathbb{H}_n(\alpha, \gamma) := \sum_{j=1}^{n} \log \phi\left(X_{t_j};\ X_{t_{j-1}} + ha_{j-1}(\alpha, \gamma),\ hc_{j-1}^2(\gamma)\right)$$

Result: [Masuda and Uehara, 2017] & [Uehara and Masuda, 2017]

Joint asymptotic normality of $(\hat{\gamma}_n, \hat{\alpha}_n)$ at speed $\sqrt{T_n}$ $\quad (T_n := nh)$

Some technical details

- Asymptotics of $\hat{\gamma}_n$ is the same as in the joint estimation, and as for $\hat{\alpha}_n$:

Explicit stochastic expansions

$$\sqrt{T_n}(\hat{\alpha}_n - \alpha_0) = \hat{\mathcal{I}}_{\alpha,n}^{-1} \left(\hat{A}_n \sqrt{T_n}(\hat{\gamma}_n - \gamma_0) + v_n \right) + o_p(1)$$

$$v_n := \frac{1}{\sqrt{T_n}} \sum_{j=1}^{n} \frac{\partial_\alpha a_{j-1}(\alpha_0, \gamma_0)}{c_{j-1}^2(\gamma_0)} (\Delta_j X - h a_{j-1}(\alpha_0, \gamma_0)),$$

$$\hat{\mathcal{I}}_{\alpha,n} := \frac{1}{n} \sum_{j=1}^{n} \frac{(\partial_\alpha \hat{a}_{j-1})^{\otimes 2}}{\hat{c}_{j-1}^2},$$

$$\hat{A}_n := \frac{1}{n} \sum_{j=1}^{n} \frac{\partial_\alpha \hat{a}_{j-1} \otimes \partial_\gamma \hat{a}_{j-1}}{\hat{c}_{j-1}^2} \xrightarrow{p} \int \frac{\partial_\alpha a(x, \alpha_0, \gamma_0) \otimes \partial_\gamma a(x, \alpha_0, \gamma_0)}{c^2(x, \gamma_0)} \pi_0(dx),$$

Asymptotic normality

$$\sqrt{T_n}\hat{\Sigma}_n^{-1/2}\begin{pmatrix}\hat{\mathcal{I}}_{\alpha,n}^{-1} & \hat{\mathcal{I}}_{\alpha,n}^{-1}\hat{A}_n \\ O & \hat{\mathcal{I}}_{\gamma,n}^{-1}\end{pmatrix}\begin{pmatrix}\hat{\alpha}_n-\alpha_0 \\ \hat{\gamma}_n-\gamma_0\end{pmatrix}\xrightarrow{\mathcal{L}} N_p(0,I)$$

- Clarifies effect of simultaneous presence of γ in the coefficients.
- $\hat{\mathcal{I}}_{\gamma,n}^{-1} := \frac{1}{n}\sum_{j=1}^{n}\frac{(\partial_\gamma \hat{c}_{j-1})^{\otimes 2}}{\hat{c}_{j-1}^2}$, and $\hat{\Sigma}_n$ is also given explicitly.
- Readily provides us with an approximate $(1-s)$-confidence set:

$$\left\{(\alpha,\gamma):\ \left|\sqrt{T_n}\hat{\Sigma}_n^{-1/2}\begin{pmatrix}\hat{\mathcal{I}}_{\alpha,n}^{-1} & \hat{\mathcal{I}}_{\alpha,n}^{-1}\hat{A}_n \\ O & \hat{\mathcal{I}}_{\gamma,n}^{-1}\end{pmatrix}\begin{pmatrix}\hat{\alpha}_n-\alpha \\ \hat{\gamma}_n-\gamma\end{pmatrix}\right|^2 \le \chi^2(p;s)\right\}$$

Put simply

- For the univariate parametric Stochastic differential equation (SDE):

$$dX_t = a(X_{t-}, \alpha)dZ_t + b(X_t, \beta)dt, \qquad \theta = (\alpha, \beta)$$

or

$$dX_t = c(X_{t-}, \gamma)dJ_t + a(X_t, \alpha, \gamma)dt, \qquad \theta = (\alpha, \gamma).$$

for stepwise estimation, we can make use of the explicit GQLF

$$\mathbb{H}_n(\theta) = \sum_{j=1}^{n} \log \phi\left(X_{t_j};\, X_{t_{j-1}} + hb_{j-1}(\beta),\, ha_{j-1}^2(\alpha)\right).$$

  - Available data $(X_{t_j})_{j=0}^{n}$; $t_j = jh_n = jh$, $T_n := nh \to \infty$, $nh^2 \to 0$.
  - Driving Lévy process s.t. $\mathbb{E}(Z_t) = 0$, $\mathrm{var}(Z_t) = t$.

- **Stability (Ergodicity)** is essential here.

- The YUIMA function `qmleLevy` was composed by Dr. Yuma Uehara.

YUIMA demo: qmleLevy

$$dX_t = a(X_t, \alpha, \gamma)dt + c(X_{t-}, \gamma)dZ_t$$

- **Usage**

```
qmleLevy(yuima, start, lower, upper, joint = FALSE, third = FALSE)
```

- **Arguments**

yuima a yuima object
lower a named list for specifying lower bounds of parameters.
upper a named list for specifying upper bounds of parameters.
start initial values to be passed to the optimizer.
joint perform joint estimation or two stage estimation?
  - by default joint=FALSE. If there exists an overlapping parameter, joint=TRUE currently does not work.

third perform third estimation?
  - by default third=FALSE. If there exists an overlapping parameter, third=TRUE currently does not work.

- **Value**

first estimated values of first estimation (scale parameters)
second estimated values of second estimation (drift parameters)
third estimated values of third estimation (scale parameters)

Example: bilateral gamma

$$dX_t = -\theta_0 X_t dt + \frac{\theta_1}{\sqrt{1+X_t^2}} dZ_t,$$

- $Z_t \sim$ Bilateral gamma$(t, \sqrt{2}, t, \sqrt{2})$, $\quad (\theta_{0,0}, \theta_{1,0}) = (1, 2)$

▷ yss2019_hm_demo.html

Example: Normal inverse Gaussian

$$dX_t = -\theta_0 X_t dt + \frac{\theta_1}{\sqrt{1+X_t^2}} dZ_t,$$

- $Z_t \sim NIG\,(\delta, 0, \delta t, 0, 1)$ with $\delta = 10$, $\quad (\theta_{0,0}, \theta_{1,0}) = (1, 2)$.

▷ yss2019_hm_demo.html

2-dim. Example: variance gamma

$$d\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} 1-\theta_0 X_{1,t} - X_{2,t} \\ -\theta_1 X_{2,t} \end{pmatrix} dt + \begin{pmatrix} \frac{\theta_2}{1+X_{1,t}^2}+1 & 0 \\ 1 & 1 \end{pmatrix} dZ_t,$$

- $Z_t \sim$ Variance gamma $\left(\frac{1}{2}t, 1, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$, $\quad (\theta_0, \theta_1, \theta_2) = (1, 2, 3)$.

▷ yss2019_hm_demo.html

2-dim. Example: Normal inverse Gaussian

$$d\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} 1-\theta_0 X_{1,t} \\ -\theta_1 X_{2,t} \end{pmatrix} dt + \begin{pmatrix} \exp\left(-\frac{\theta_2}{1+X_{1,t}^2}\right) & 0 \\ 1 & \exp\left(-\frac{\theta_3}{\sqrt{1+X_{2,t}^2}}\right) \end{pmatrix} dZ_t,$$

- $Z_t \sim NIG_2\left(\frac{1}{\sqrt{\pi}}t, 1, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad (\theta_0, \theta_1, \theta_2, \theta_3) = (1, 2, 3, 4).$

▷ yss2019_hm_demo.html

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