YUIMA SUMMER SCHOOL Brixen (June 26)

Lecture 5
Refresh on diffusion processes

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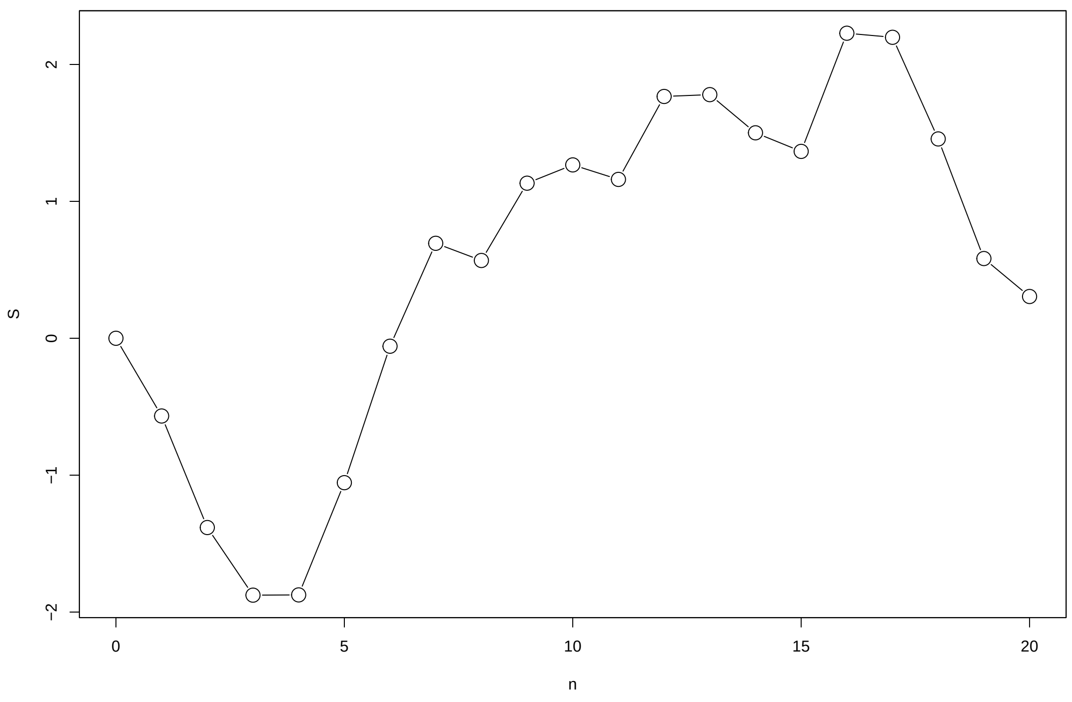
Recall Wiener process (Brownian motion)

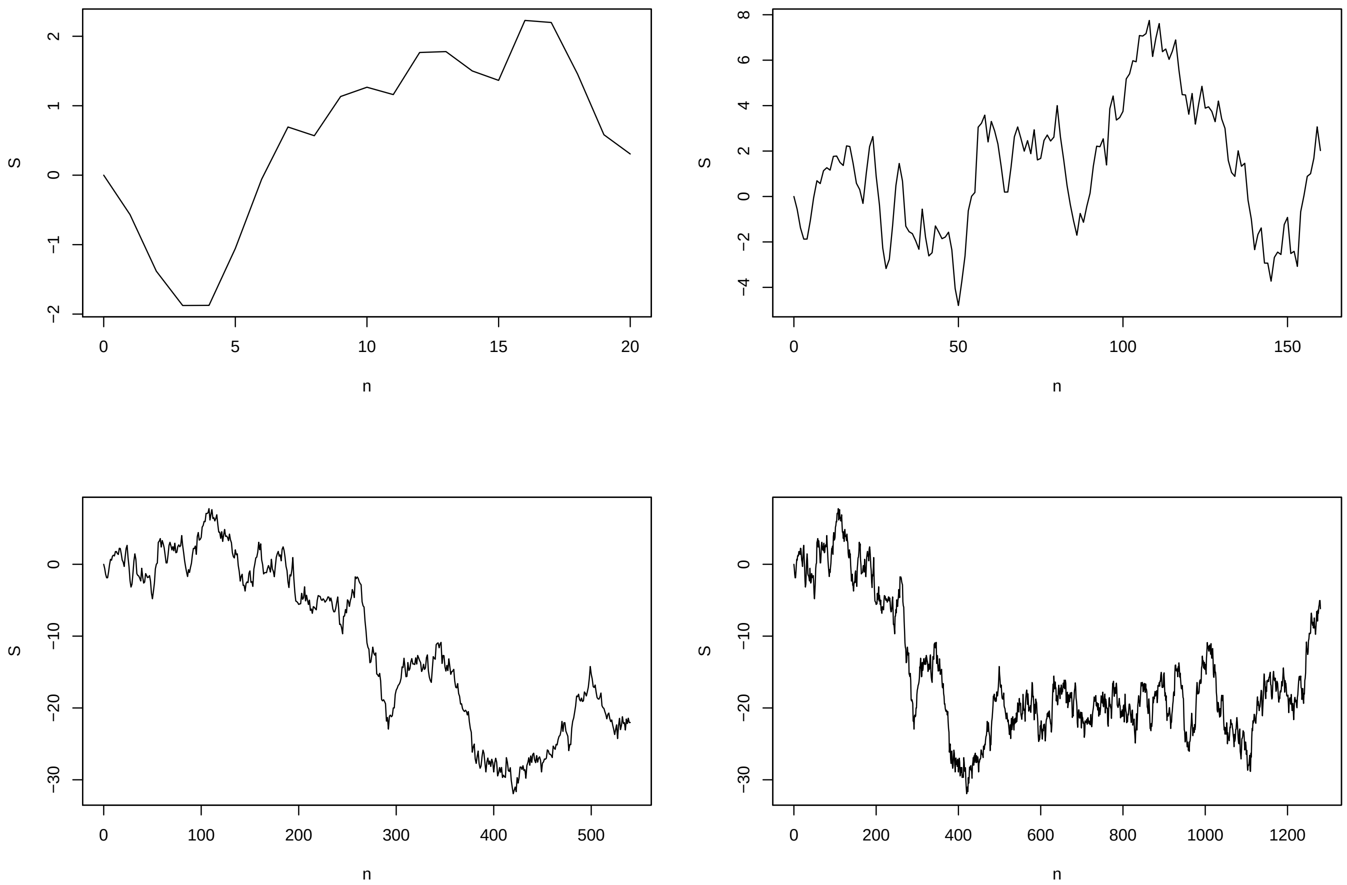
Random walk

- $(\xi_j)_{j\in\mathbb{N}}$: independent and identically distributed random variables (defined on a probability space $(\Omega, \mathcal{F}, P)$).
- $S_0 := 0,\ S_n := \sum_{j=1}^{n} \xi_j$



Random walk

- **Take more data and zoom out**



- **What appears when $n = \infty$?**

Donsker's theorem (Functional Central Limit Theorem)

- $(\xi_j)_{j\in\mathbb{N}}$: independent and identically distributed random variables with $E[\xi_1]=0$, $E[\xi_1^2]=1$, and $S_0=0$, $S_n=\sum_{j=1}^n \xi_j$, as before
- scaling space-time and linear interpolation

$$X_t^n = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} + \frac{(nt-\lfloor nt \rfloor)}{\sqrt{n}} \xi_{\lfloor nt \rfloor + 1}$$

Theorem 1.

$$X^n \to^d W \quad \text{in } C \quad (n\to\infty)$$

- C is the space of continuous functions $f:\mathbb{R}_+\to\mathbb{R}$, equipped with a certain topology. $\mathbb{R}_+=[0,\infty)$.
- A standard Wiener process (Brownian motion) W appeared as the limit.

Wiener process (Brownian motion)

- $W = (W_t)_{t \in \mathbb{R}_+}$: a standard Wiener process
- W has the following properties.

Standard Wiener process

(i) $W_0 = 0$.

(ii) $W_t - W_s \sim N(0, t-s)$ for $s < t$.

(iii) For $0 = t_0 < t_1 < \cdots < t_k$ ($k \in \mathbb{N}$), the random variables

$$W_{t_1},\ W_{t_2} - W_{t_1},\ \cdots,\ W_{t_k} - W_{t_{k-1}}$$

are independent.

(iv) Each path $t \mapsto W_t$ is continuous.

- The Wiener process has many interesting properties.

Wiener integral

Transform of a Wiener process

- simple function $h_s = \sum_{j=1}^{k} a_{j-1} \mathbf{1}_{(t_{j-1},t_j]}(s)$, $a_0, \ldots, a_{k-1} \in \mathbb{R}$ and $0 = t_0 \le t_1 \le \cdots \le t_k$
- $$J(h)_t := \sum_{j=1}^{k} a_{j-1}\big(W_{t_j \wedge t} - W_{t_{j-1} \wedge t}\big),\ s \wedge t = \min\{s,t\}$$

Exercise 1. (a) $E[J(h)_t] = 0$

(b) $E[J(h)_t^2] = \|h\|^2_{L^2([0,t],ds)}$, where $\|h\|_{L^2([0,t],ds)} = \big(\int_0^t h_s^2 ds\big)^{1/2}$. [For the simple function, the value is $\sum_{j=1}^{k} a_{j-1}^2 (t_j \wedge t - t_{j-1} \wedge t)$.]

(c) $J(h)_t \sim N\big(0, \|h\|^2_{L^2([0,t],ds)}\big)$.

(d) For $s < t$, the conditional expectation

$$E\big[J(h)_t \big| W_r \ (r \le s)\big] = J(h)_s.$$

Wiener integral

- As a limit of $J(H)_t$ discussed above, we obtain the so-called Wiener integral.

If $h : \mathbb{R}_+ \to \mathbb{R}$ is a locally bounded piecewise continuous function, then the Wiener integral

$$J(h)_t = \int_0^t h_s \, dW_s$$

is well defined and the properties in Exercise 1 hold.

- Some details are given below.

Wiener integral

- $(\Omega, \mathcal{F}, P)$: a probability space
- $L^2(P)$: the space of square-integrable random variables. $\|X\|_2 = \{ \int_\Omega X(\omega)^2 P(d\omega) \}^{1/2}$.
- Fact. For any $h \in L^2(\mathbb{R}_+, ds)$, there exists a sequence of simple functions $h^n_\cdot$ such that $\|h^n - h\|_{L^2([0,t],ds)} \to 0$ as $n \to \infty$ for every $t \in \mathbb{R}_+$.
- The property (b) gives an isometry

$$\|J(h)_t\|_2 = \|h\|_{L^2([0,t],ds)}$$

- Thus, we can define $J(h)_t = \lim_{n\to\infty} J(h^n)_t$ in $L^2(P)$.
- We write

$$J(h)_t = \int_0^t h_s dW_s$$

Wiener integral

- More simply, as already stated, e.g.,

  If $h : \mathbb{R}_+ \to \mathbb{R}$ is a locally bounded piecewise continuous function, then the Wiener integral

$$J(h)_t = \int_0^t h_s \, dW_s$$

  is well defined and the properties in Exercise 1 hold.

Exercise 2. Verify

$$\int_0^t e^{\theta s} dW_s = J(e^{\theta \cdot})_t \sim N\big(0, (e^{2\theta t} - 1)/(2\theta)\big)$$

for $\theta \neq 0$.

Itô integral

Itô integral

- We say that a random variable ξ is <u>$\mathcal{F}_t$-measurable</u> if ξ is a function of $(W_s)_{s\in[0,t]}$.
- $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is called a <u>filtration</u>.
- We say a stochastic process $X = (X_t)_{t\in\mathbb{R}_+}$ is <u>adapted</u> to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if X_t is $\mathcal{F}_t$-measurable for every $t \in \mathbb{R}_+$.
- Remark. More generally and rigorously, $\mathrm{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a <u>filtration</u>, i.e., each $\mathcal{F}_t$ is a sub σ-field of $\mathcal{F}$, and $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. Usually, the right-continuity $\mathcal{F}_{t+} = \cap_{u>t}\mathcal{F}_u = \mathcal{F}_t$ is assumed. We suppose that $W = (W_t)_{t\in\mathbb{R}_+}$ is adapted to F, and that $W_t - W_s$ is independent of $\mathcal{F}_s$ for $s < t$.

Itô integral

- We call a stochastic process $H = (H_t)_{t\in\mathbb{R}_+}$ a simple predictable process (with respect to $\mathbf{F}$) if it admits a representation

$$H_t = H_0 1_{\{0\}}(t) + \sum_{j=1}^{J} H_{(j-1)} 1_{(t_{j-1},t_j]}(t) \qquad (1)$$

for some $J \in \mathbb{Z}_+$, $0 = t_0 \leq t_1 \leq \cdots \leq t_J$, some $\mathcal{F}_0$-measurable random variable H_0 and some $\mathcal{F}_{t_j}$-measurable random variable $H_{(j)}$.

- Define $J_W(H) = (J_W(H)_t)_{t\in\mathbb{R}_+}$ by

$$J_W(H)_t = \sum_{j=1}^{J} H_{(j-1)}\big(W_{t_j\wedge t} - W_{t_{j-1}\wedge t}\big)$$

Itô integral

- Denote by $\mathbb{S}_b$ the set of all bounded simple predictable processes.
- For $H \in \mathbb{S}_b$,

$$E\left[J_W(H)_t^2\right] = E\left[\int_0^t H_s^2 ds\right] \tag{2}$$

Exercise 3. Verify (2).

Hint. When $t_{j-1} < t_{j'-1} \le t$, we have

$$
\begin{aligned}
& E\left[H_{(j-1)}H_{(j'-1)}\left(W_{t_j \wedge t} - W_{t_{j-1}\wedge t}\right)\left(W_{t_{j'}\wedge t} - W_{t_{j'-1}\wedge t}\right)\right] \\
= & E\Bigg[H_{(j-1)}H_{(j'-1)}\left(W_{t_j\wedge t} - W_{t_{j-1}\wedge t}\right) \\
& \times E\left[W_{t_{j'}\wedge t} - W_{t_{j'-1}\wedge t} | \mathcal{F}_{t_{j'-1}}\right]\Bigg] \\
= & 0.
\end{aligned}
$$

Itô integral

- $\mathbb{L}$: the set of F-adapted processes that are left-continuous and admitting right-hand limits.
- Starting with the isometry (2), it is possible to extend J_W from $\mathbb{S}_b$ to $\mathbb{L}$ (with some topology). Therefore we can define the Itô integral $J_W(H)_t$ for $H \in \mathbb{L}$. We write

$$J_W(H)_t = \int_0^t H_s \, dW_s$$

- Then, J_W is extending the Wiener integral J.
- $t \mapsto J_W(H)$ is continuous (such version exists)
- Martingale property.

$$E\big[J_W(H)_t | \mathcal{F}_s\big] = J_W(H)_s \quad a.s. \quad (s < t)$$

if $E[\int_0^t H_s^2 ds] < \infty$.

Itô integral

- L^p**-maximum inequality. For** $p \in (1, \infty)$,

$$\left\| \sup_t |J_W(H)_t| \right\|_p \leq \frac{p}{p-1} \sup_t \|J_W(H)_t\|_p$$

- **Burkholder-Davis-Gundy inequality. For** $p \in (0, \infty)$, **there exist constants** A_p **and** B_p **such that**

$$\begin{aligned} A_p \| [J_W(H), J_W(H)]_t^{1/2} \|_p &\leq \left\| \sup_{s \in [0,t]} |J_W(H)_s| \right\|_p \\ &\leq B_p \| [J_W(H), J_W(H)]_t^{1/2} \|_p \end{aligned}$$

where $[J_W(H), J_W(H)]_t = \int_0^t H_s^2 ds$.

Itô's formula

Itô's formula, starting with a simple example

- $f \in C_b^2(\mathbb{R})$, $t_j = j/n$
- $M_n := \sum_{j=1}^n f''(W_{t_{j-1}})\{(W_{t_j} - W_{t_{j-1}})^2 - n^{-1}\}$
- Then

$$\begin{aligned} E[M_n^2] &= 2n^{-2}\sum_{j=1}^n E\left[f''(W_{t_{j-1}})^2\right] \\ &\leq 2\|f''\|_\infty^2 n^{-1} \to 0 \quad (n \to \infty) \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{j=1}^n f''(W_{t_{j-1}})(W_{t_j} - W_{t_{j-1}})^2 \\ = &\sum_{j=1}^n f''(W_{t_{j-1}})n^{-1} + o_p(1) \to^p \int_0^1 f''(W_t)dt \end{aligned}$$

Itô's formula

- **In this situation,**

$$
\begin{aligned}
& f(W_1) - f(W_0) \\
= \; & \sum_{j=1}^{n} \big(f(W_{t_j}) - f(W_{t_{j-1}}) \big) \\
= \; & \sum_{j=1}^{n} f'(W_{t_{j-1}})(W_{t_j} - W_{t_{j-1}}) \\
& + \frac{1}{2} \sum_{j=1}^{n} f''(W_{t_{j-1}})(W_{t_j} - W_{t_{j-1}})^2 + o_p(1) \\
\to^p \; & \int_0^1 f'(W_t) dW_t + \frac{1}{2} \int_0^1 f''(W_t) dt \quad (n \to \infty)
\end{aligned}
$$

Itô's formula

- Consequently,

$$f(W_1) = f(W_0) + \int_0^1 f'(W_t)dW_t + \frac{1}{2}\int_0^1 f''(W_t)dt.$$

- More generally,
  Itô's fomula:

$$\begin{aligned} f(X_t) &= f(X_r) + \int_r^t f'(X_s)dX_s + \frac{1}{2}\int_r^t f''(X_s)H_s^2 ds \\ &= f(X_r) + \int_r^t f'(X_s)K_s ds + \int_r^t f'(X_s)H_s dW_s \\ &\quad + \frac{1}{2}\int_r^t f''(X_s)H_s^2 ds \quad (0 \le r \le t) \end{aligned}$$

  for $X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$ with $K, H \in \mathbb{L}$ and $f \in C^2(\mathbb{R})$ (boundedness is not necessary).

Stochastic differential equation

Stochastic integral equation

- F $= (\mathcal{F}_t)_{t\in\mathbb{R}_+}$, a filtration on a probability space $(\Omega, \mathcal{F}, P)$.
- A stochastic integral equation

$$X_t = x_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dW_s \qquad (3)$$

where a and b are given functions.

- An F-adapted continuous process $X = (X_t)_{t\in\mathbb{R}_+}$ for which the equality (3) holds for every t is called a solution to the stochastic integral equation. The last term of the right-hand side of (3) is the Itô integral.

Stochastic integral equation

- Given W, the equation (3) has a unique solution X if the following condition is satisfied:
  $\exists L > 0$ such that

  $$|a(x) - a(y)| + |b(x) - b(y)| \leq L|x - y| \quad (x, y \in \mathbb{R})$$

- This solution is called a strong solution.
- Equation (3) is equivalently expressed as a stochastic differential equation

$$\begin{cases} dX_t = a(X_t)dt + b(X_t)dW_t \\ X_0 = x_0 \end{cases} \tag{4}$$

Diffusion process

- A continuous-time strongly Markovian process with almost surely continuous sample paths is called a diffusion process.
- Many diffusion processes can be constructed by a stochastic differential equation having a unique solution.
- The generator gives essential information of the diffusion process. For the diffusion corresponding to the stochastic differential equation (4), the generator is given by

$$Lf = a(x)f'(x) + \frac{1}{2}b(x)^2 f''(x)$$

for $f \in C^2(\mathbb{R})$.

Diffusion process

- **For a solution X to (4), Itô's formula is written as**

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)b(X_s)dW_s + \int_0^t Lf(X_s)ds.$$

Diffusion process: an example

- Let

$$X_t = x_0 \exp\left((\mu - 2^{-1}\sigma^2)t + \sigma W_t\right)$$

- By applying Itô's formula to the function $f(x) = x_0 e^x$ and the process $(\mu - 2^{-1}\sigma^2)t + \sigma W_t$, we obtain

$$\begin{aligned} X_t &= x_0 + \int_0^t X_s(\mu - 2^{-1}\sigma^2)ds + \int_0^t X_s \sigma dW_s \\ &\quad + \frac{1}{2}\int_0^t X_s \sigma^2 ds \\ &= x_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s \quad \text{equivalently} \end{aligned}$$

- Geometric Brownian motion (Black-Scholes model)

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \qquad X_0 = x_0$$

Realized volatility

- $t_j = t_j^n = j/n$.
- $V_n := \sum_{j=1}^{n} \left(X_{t_j} - X_{t_{j-1}}\right)^2$
- Then $V_n \to^p V_\infty = \int_0^1 H_s^2 ds$ as $n \to \infty$
  for $X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$ with $K, H \in \mathbb{L}$.

Exercise 4. Prove this fact for the geometric Brownian motion analytically and/or by simulation with YUIMA.

Euler-Maruyama approximation

- **An approximation to the solution of the SDE (4) is given by the so-called Euler-Maruyama method.**
- $t_j = t_j^n := jt/n$ **for given** $t > 0$. $h = h_n := t/n$
- Y_{t_j} **is recursively generated by**

$$Y_{t_j} = Y_{t_{j-1}} + a(Y_{t_{j-1}})h + b(Y_{t_{j-1}})h^{1/2}\xi_j, \quad Y_0 = x_0$$

with $\xi_j \sim$ **i.i.d.** $N(0,1)$.

- Y_{t_j} **approximates** X_{t_j} **when** n **is large.**
- **The YUIMA simulate basically uses this approximation method.**

Multi-dimensional processes

- $(\Omega, \mathcal{F}, P)$: a probability space equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$
- an r-dimensional Wiener process $W = (W_t^\alpha)_{\alpha=1,\ldots,\mathsf{r},\, t\in\mathbb{R}_+}$: each $W^\alpha = (W_t^\alpha)_{t\in\mathbb{R}_+}$ is a standard Wiener process and $W^1, \ldots, W^{\mathsf{r}}$ are independent.
- Multi-dimensional Itô process $X = (X^i)_{i=1,\ldots,\mathsf{d}}$

$$X_t^i = X_0^i + \int_0^t K^i(s)dt + \sum_{\alpha=1}^{\mathsf{r}} \int_0^t H_\alpha^i(s)dW_s^\alpha \quad (i = 1, \ldots, \mathsf{d})$$

for K^i, $H_\alpha^i \in \mathbb{L}$ such that

$$\sum_{i,\alpha} \int_0^t (|K^i(s)| + |H_\alpha^i(s)|^2)ds < \infty \quad (t \in \mathbb{R}_+)$$

Multi-dimensional processes

- **Itô's formula**

$$\begin{aligned} f(X_t) &= f(X_0) \\ &\quad + \int_0^t \Big\{ \sum_i \partial_i f(X_s) K^i(s) ds + \frac{1}{2} \sum_{i,j,\alpha} \partial_i \partial_j f(X_s) H^i_\alpha(s) H^j_\alpha(s) \Big\} ds \\ &\quad + \sum_{i,\alpha} \int_0^t \partial_i f(X_s) H^i_\alpha(s) dW^\alpha_s \end{aligned}$$

for $f \in C^2(\mathbb{R}^{\mathsf{d}})$, where $\partial_i = \partial / \partial x^i$ for $x = (x^i)_{i=1,\dots,\mathsf{d}}$.

Multi-dimensional processes

- Multi-dimensional SDEs are also considered.
- e.g. Heston model

$$\begin{cases} dX_t^1 = \mu X_t^1 dt + \sqrt{X_t^2} X_t^1 dB_t^1 \\ dX_t^2 = \kappa(\theta - X_t^2) dt + \epsilon \sqrt{X_t^2} dB_t^2 \end{cases}$$

where $B^1 = W^1$ and $B^2 = \rho W^1 + \sqrt{1 - \rho^2} W^2$.