Refresh on probability & stochastic processes

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1. Mode of convergence
   - Example: i.i.d. obsevations
2. Stochastic processes and Brownian motion
   - Stochastic processes
   - Brownian motion
3. Brief review on Itô calculus
   - Itô integral
   - Itô's formula

Mode of convergence

- $U_1, U_2, \ldots$: sequence of random vectors (of a common dimension)
- U_∞: a random vector (of the same dimension as U_i's)
- $(U_n)_{n=1}^\infty$ is said to **converge in probability** to U_∞ if

$$P(\|U_n - U_\infty\| > \varepsilon) \to 0$$

  as $n \to \infty$ for any $\varepsilon > 0$ ($\|\cdot\|$ denotes the Euclidean norm); we then write $U_n \to^p U_\infty$
- $(U_n)_{n=1}^\infty$ is said to **converge in law** to U_∞ if

$$f(U_n) \to f(U_\infty)$$

  as $n \to \infty$ for any continuous bounded function f; we then write $U_n \to^d U_\infty$

- <u>Statistical context</u>
  - "$U_n \to^p U_\infty$" means that U_∞ is an asymptotically good estimator for U_∞ (**consistent** estimator for U_∞)
  - If "$U_n \to^d U_\infty$", we can approximate the distribution of U_n by U_∞'s: In the univariate case,

$$P(a \leq U_n \leq b) \to P(a \leq U_\infty \leq b)$$

    as $n \to \infty$ for any real numbers a, b such that $P(U_\infty = a) = P(U_\infty = b) = 0$

- **Continuous mapping theorem**: If f is a continuous function,

$$U_n \to^p U_\infty \Rightarrow f(U_n) \to^p f(U_\infty)$$

  and

$$U_n \to^d U_\infty \Rightarrow f(U_n) \to^d f(U_\infty)$$

Example: i.i.d. obsevations

- $X_1, X_2, \ldots$: i.i.d. random variables with mean μ and variance σ^2
- $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$: sample mean
- **Law of Large Numbers** (LLN, consistency of $\bar{X}_n$):

$$\bar{X}_n \to^p \mu \qquad (n \to \infty)$$

- **Central Limit Theorem** (CLT, asymptotic normality of $\bar{X}_n$):

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \to^d Z \qquad (n \to \infty),$$

where Z is a standard normal variable

Example: i.i.d. obsevations

- Reproducing LLN and CLT by R
- Assume X_i's are Bernoulli random variables with success probability p:

$$P(X_i = 1) = 1 - P(X_i = 0) = p$$

  - X_i has mean p and variance $p(1-p)$
  - $\sum_{i=1}^{n} X_i$ follows the binomial distribution with n trials and success probability p
- **R examples**: `lln.r`, `clt.r`

Stochastic processes

- **(Continuous-time) stochastic process**
  - Randomly determined function of time t
  - The time t usually varies in the closed interval $[0, T]$ ($T > 0$) or the half non-negative real line $[0, \infty)$

Example 1

Let ϵ be a random variable. For every $t \in [0, T]$, set

$$X_t := \sin(\epsilon t).$$

Then we obtain a stochastic process $(X_t)_{t\in[0,T]}$.

Stochastic processes

- When we say "$(X_t)_{t\in[0,T]}$ is a stochastic process", X_t represents the function value of the stochastic process at the time t
  - Since the function is randomly determined, so is X_t, and thus X_t is a random variable
- Each random function possibly realized by a stochastic process $X = (X_t)_{t\in[0,T]}$ is called a **sample path** of X
- When we simulate a stochastic process, since it is impossible to continuously vary the time t in practice, we proceed as follows:
  1. We fix (sufficiently fine) sampling times in the time interval $[0, T]$
  2. We simulate sample paths of the stochastic process evaluated at those sampling times

Stochastic processes

- The sampling times are usually taken to be equi-spaced, so we focus only on such a case in the following
- Hence, the sampling times are of the form $t_i = Ti/n$ $(i = 0, 1, \dots, n)$, and we consider the number n as a parameter of simulation
- Here we give an example of an R code to simulate the stochastic process given by Example 1
- **R example**: `process.r`

Brownian motion

Definition 1 (Brownian motion)

A stochastic process $(B_t)_{t\in[0,T]}$ is said to be a **(standard) Brownian motion** if it satisfies the following properties:

(i) $B_t = 0$ and $\mathrm{E}[B_t^2] = t$ for all $t \in [0, T]$.

(ii) $(B_t)_{t\in[0,T]}$ has continuous sample paths

(iii) (independence of increments) For any $0 \le t_0 < t_1 < \cdots < t_n \le T$, $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent random variables

(iv) (stationarity of increments) For any $0 \le s < t \le T$, $B_t - B_s$ has the same law as B_{t-s}

Brownian motion is also called **Wiener process**

Normality of increments of Brownian motion

- The definition of Brownian motion seems to impose no restriction on the laws of the random variables B_t
- However, this definition indeed implies that B_t should follow a normal distribution!

Proposition 1 (Normality of increments of Brownian motion)

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion. Then, for any $0 \leq s < t \leq T$, $B_t - B_s$ follows the normal distribution with mean 0 and variance $t - s$.

Simulation of Brownian motion

- Using Proposition 1 and the independence of increments of Brownian motion, we can simulate sample paths of Brownian motion at the sampling times $t_i = Ti/n$ ($i = 0, 1, \dots, n$) as follows:
  1. Generate n independent standard normal random variables $Z_1, Z_2, \dots, Z_n$.
  2. Set $B_{t_0} := 0$ and
  $$B_{t_i} := (Z_1 + Z_2 + \cdots + Z_n)\sqrt{\Delta_n}$$
  for $i = 1, 2, \dots, n$, where $\Delta_n := T/n$.
- **R example**: `bm.r`

Brief review on Itô calculus

- From now on, we briefly review **Itô calculus**, a standard mathematical tool to analyze continuous-time stochastic processes (especially Brownian motion driven models)
- In the following, $B = (B_t)_{t \in [0,T]}$ denotes a Brownian motion

Itô integral

- Let $X = (X_t)_{t\in[0,T]}$ and $Y = (Y_t)_{t\in[0,T]}$ be two stochastic processes having continuous sample paths
- Our aim here is to properly define an "integral" of the form

$$\int_0^t X_s dY_s, \qquad t \in [0, T] \tag{1}$$

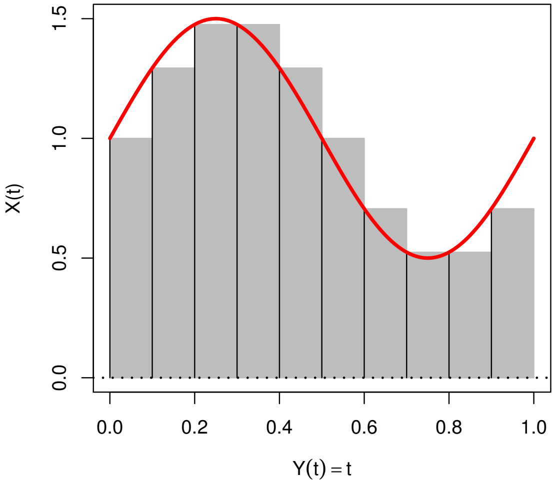
- In particular, we would like to define the above form of integral when $Y = B$, i.e. the integrator is Brownian motion

Itô integral

- A "standard" approach to define an integral of the form (1):
  (i) Divide the interval $[0, t]$ into n equi-spaced time points $t_{n,i} := it/n$ $(i = 0, 1, \ldots,)$
  (ii) If the "Riemann sum"

$$\sum_{i=1}^{n} X_{t_{n,i-1}}(Y_{t_{n,i}} - Y_{t_{n,i-1}}) \tag{2}$$

  converges as $n \to \infty$ in *some sense*, we define the integral (1) by the limit variable
- Problem When does the Riemann sum (2) have the limit?
  - It is well-known that it has the limit in the usual sense if Y has differentiable sample paths
  - However, sample paths of Brownian motion are NOT differentiable almost everywhere!


X(t)
Y(t) = t
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0

Itô integral

- If we consider the Brownian motion B as the "integrator" process Y, we need an additional restriction on the "integrated" process X to ensure that the Riemann sum (2) has the limit
- Specifically, it is known that the following condition is sufficient to ensure the existence of such a limit:
  - For every $t \in [0, T]$, X_t is a functional of $(B_s)_{s \in [0,t]}$:

    $$X_t = f_t((B_s)_{s \in [0,t]}).$$

    Here, f_t is a functional of functions on $[0, t]$ with satisfying some regularity conditions (w.r.t. measurability)
- Such a stochastic process X is said to be **adapted** (w.r.t. the filtration generated by B)
  - Ex. If X is given by $X_t = f(B_t)$ $(t \in [0, T])$ with some continuous function $f : \mathbb{R} \to \mathbb{R}$, then X is adapted

Itô integral

- If X is an adapted process having continuous sample paths, the Riemann sum
$$\sum_{i=1}^{n} X_{t_{n,i-1}}(B_{t_{n,i}} - B_{t_{n,i-1}})$$
has the limit as $n \to \infty$ (in the sense of convergence in probability)
- This limit is denoted by
$$\int_0^t X_s dB_s \tag{3}$$
and called the **Itô integral** or **stochastic integral** of X w.r.t. B

Itô integral vs backward Itô integral

- What happens if we consider the following form of Riemann sum?

$$\sum_{i=1}^{n} X_{t_{n,i}}(B_{t_{n,i}} - B_{t_{n,i-1}}) \tag{4}$$

- If sample paths of B *were* differentiable, this should converge to $\int_0^t X_s dB_s$ as $n \to \infty$; but this is NOT the case in general!
- For example, if $X = B$, (4) converges to

$$\int_0^t B_s dB_s + t$$

- The limit of the Riemann sum (4) is called the **backward Itô integral** of X w.r.t. B if it exists
- Let us check this phenomenon by R: `backward.r`

Itô's formula

- Suppose that X has differentiable sample paths
- Then, for any C^1 function $f : \mathbb{R} \to \mathbb{R}$, we have

$$f(X_t) - f(X_0) = \int_0^t \frac{d}{ds} f(X_s) ds = \int_0^t f'(X_s) \frac{d}{ds} X_s ds$$
$$= \int_0^t f'(X_s) dX_s$$

  by the fundamental theorem of calculus and chain rule
- Thus, we obtain an integral representation of $f(X_t)$

Itô's formula

- Alternatively, this identity can be viewed as a way to *directly* compute the integral
$$\int_0^t f'(X_s)dX_s$$
by $f(X_t) - f(X_0)$ rather than to approximate it by the corresponding Riemann sums
- For this reason, it would be convenient if we have an analogous formula for the case of Itô integrals
- However, the above argument is not applicable to Itô integrals because sample paths of Brownian motion are not differentiable
- Indeed, we need an additional term to get an analogous formula for the case of Itô integrals as follows:

Itô's formula

Theorem (Itô's formula for Brownian motion)

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then we have

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

Itô's formula: Examples with R codes

- **Example 1** Taking $f(x) = x^2$, we have $f'(x) = 2x$ and $f''(x) = 2$, so we obtain
$$B_t^2 = 2\int_0^t B_s dB_s + t.$$
- **Example 2** Taking $f(x) = e^x$, we have $f'(x) = f''(x) = e^x$, so we obtain
$$e^{B_t} = 1 + \int_0^t e^{B_s} dB_s + \frac{1}{2}\int_0^t e^{B_s} ds.$$
- **R examples**: `ito-1.r`, `ito-2.r`