Simulation for diffusion processes

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Stochastic Differential Equation (SDE)

- $a(t,x), b(t,x)$: functions of time t and state x
- $B = (B_t)_{t \in [0,T]}$: Brownian motion
- An equation w.r.t. the unknown stochastic process $X = (X_t)_{t \in [0,T]}$ of the form

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dB_s, \quad t \in [0, T] \qquad (1)$$

is called **Stochastic Differential Equation (SDE)** driven by B

Stochastic Differential Equation (SDE)

- (1) is an integral equation rather than a differential equation, but the name "SDE" comes from its introduction by K. Itô
- In particular, (1) is customarily expressed as the following differential form:

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t, \quad t \in [0, T]. \tag{2}$$

- The functions $a(t,x), b(t,x)$ are often referred to as the **drift** and **diffusion** coefficients of SDE (2), respectively
- Here, we aim at presenting representative stochastic processes modeled as SDEs and simulating them by `yuima`

Euler-Maruyama scheme

- We can rarely have an explicit solution of a SDE
- Hence, we need to numerically simulate a solution of the SDE instead
- Here, we present the **Euler-Maruyama scheme**, the most popular method to simulate a solution of a given SDE

Euler-Maruyama scheme

- We consider the problem of simulating a solution of SDE (2) on the equi-spaced time points $t_i = i\Delta_n$ ($i = 0, 1, \ldots, n$; $\Delta_n = T/n$) on the time interval $[0, T]$
- Equation (1) yields

$$X_{t_{i+1}} - X_{t_i} = \int_{t_i}^{t_{i+1}} a(s, X_s)ds + \int_{t_i}^{t_{i+1}} b(s, X_s)dB_s, \quad i = 1, \ldots, n$$

- If n is sufficiently large, the right hand side of the above equation would be well-approximated by

$$a(t_i, X_{t_i})\Delta_n + b(t_i, X_{t_i})(B_{t_{i+1}} - B_{t_i})$$

Euler-Maruyama scheme

- Based on this consideration, we approximate the series $(X_{t_i})_{i=0}^n$ by the following one: We define the series $(\widehat{X}_{n,t_i})_{i=0}^n$ by $\widehat{X}_{n,0} := X_0$ and

$$\widehat{X}_{n,t_{i+1}} = \widehat{X}_{n,t_i} + a(t_i, \widehat{X}_{n,t_i})\Delta_n + b(t_i, \widehat{X}_{n,t_i})(B_{t_{i+1}} - B_{t_i}), \quad i = 1, \dots, n$$

- Under some regularity assumptions, we can prove

$$\lim_{n\to\infty} \max_{i=0,1\dots,n} |\widehat{X}_{n,t_i} - X_{t_i}| = 0,$$

which validates the above approximation for sufficiently large n

Simulation of SDEs by YUIMA

- The R package `yuima` provides a framework to systematically implement simulation, estimation and model evaluation of general SDEs by intuitive description
- As an illustration, we present how to simulate the following SDE (containing an unknown parameter θ) by `yuima`

$$dX_t = -\theta X_t + \frac{1}{1+X_t^2} dB_t, \qquad t \in [0,1]$$

- **R example**: `simulation-ex.r`

Some examples: Complicated coefficients

- Let us consider the following SDE:

$$dX_t = \mu X_t dt + \sigma(X_t) dB_t, \qquad t \in [0, T] \tag{3}$$

where

$$\sigma(x) = \begin{cases} \sigma_1 x & \text{if } x < \tau, \\ \sigma_2 x & \text{if } x \geq \tau, \end{cases}$$

and $\mu \in \mathbb{R}$ and $\sigma_1, \sigma_2, \tau > 0$ are constants

- This is a variant of the famous Black-Scholes model (a.k.a. geometric Brownian motion), but the "volatility" coefficient depends on "regimes"
- How do we simulate it in YUIMA?
- **R example**: `gbm-regime.r`

Some examples: Random initial condition

- We can specify random initial conditions with unknown parameters in the yuima
- As an illustration, we consider the following SDE:

$$dX_t = -\theta(X_t - \mu)dt + \sigma\sqrt{X_t}dB_t, \tag{4}$$

  where $\mu, \sigma \geq 0$ and $\theta \in \mathbb{R}$ are constants
- This is known as the **Cox-Ingersoll-Ross (CIR) model**
- To ensure the solution satisfies $X_t > 0$ a.s. for all t, we need to impose $X_0 > 0$ a.s. and

$$2\theta\mu \geq \sigma^2,$$

  where the latter is known as the **Feller condition**

Some examples: Random initial condition

- The CIR model has the gamma distribution $\mathrm{Gamma}(2\theta\mu/\sigma^2, 2\theta/\sigma^2)$ with shape $2\theta\mu/\sigma^2$ and rate $2\theta/\sigma^2$ as the stationary distribution
- Namely, if X_0 is drawn from $\mathrm{Gamma}(2\theta\mu/\sigma^2, 2\theta/\sigma^2)$, then X_t follows $\mathrm{Gamma}(2\theta\mu/\sigma^2, 2\theta/\sigma^2)$ for all t
- Let us check this fact by simulation in YUIMA!
- **R example**: `random-init.r`

Some examples: Time-inhomogeneous diffusion

- It is of course possible to include the time variable in the drift and/or diffusion coefficients
- For example, let us simulate the following SDE:

$$dX_t = (t - X_t)dt + X_t(1 - t^{1/3} + 0.5t^2)dB_t, \qquad X_0 = 1$$

- **R example**: `inhomogeneous.r`

Multi-dimensional process

- Thus far, we treat only univariate stochastic processes/SDEs, but they can be naturally extended to the multivariate case
- The Euler-Maruyama scheme can be naturally extended to the multivariate case, so we can simulate multivariate SDEs by utilizing it as well
- The function `simulate()` of the package `yuima` supports simulation of multivariate SDEs
- We give some examples of how to simulate multivariate SDEs in YUIMA rather than discuss general aspects of such models

Multi-dimensional process

- Let us consider the following two-dimensional SDE:

$$\begin{cases} dX_{1,t} &= -3X_{1,t}dt + dB_{1,t} + X_{2,t}dB_{3,t}, \\ dX_{2,t} &= -(X_{1,t} + 2X_{2,t})dt + X_{1,t}dB_{1,t} + 3dB_{2,t}, \end{cases} \tag{5}$$

  where $(B_{1,t})_{t\in[0,T]}$, $(B_{2,t})_{t\in[0,T]}$, $(B_{3,t})_{t\in[0,T]}$ are three independent Brownian motions

- Our aim is to simulate sample paths of the two-dimensional process $X_t = (X_{1,t}, X_{2,t})$, $t \in [0, T]$ by the Euler-Maruyama scheme

Multi-dimensional process

- In the package `yuima`, multi-dimensional SDEs are expressed in vector forms, so we rewrite SDE (5) in the vector form as follows:

$$\begin{pmatrix} dX_{1,t} \\ dX_{2,t} \end{pmatrix} = \begin{pmatrix} -3X_{1,t} \\ -X_{1,t} - 2X_{2,t} \end{pmatrix} + \begin{pmatrix} 1 & 0 & X_{2,t} \\ X_{1,t} & 3 & 0 \end{pmatrix} \begin{pmatrix} dB_{1,t} \\ dB_{2,t} \\ dB_{3,t} \end{pmatrix}$$

- **R example**: `simulate-multi.r`

Comparing the same models with different parameters

- Taking a single Brownian motion across all the components, we can examine the effects of parameters on the sample path
- For example, let us consider the following four CIR models:

$$\begin{cases} dX_{1,t} &= -(X_{1,t} - 1)dt + \sqrt{X_{1,t}}dB_t, \\ dX_{2,t} &= -(X_{2,t} - 5)dt + \sqrt{X_{2,t}}dB_t, \\ dX_{3,t} &= -5(X_{3,t} - 1)dt + \sqrt{X_{3,t}}dB_t, \\ dX_{4,t} &= -(X_{4,t} - 1)dt + \frac{1}{5}\sqrt{X_{4,t}}dB_t, \end{cases}$$

- These CIR models share a common Brownian motion but have different parameters, so we can compare the effects of the different parameters with a common random seed
- **R example**: `different-param.r`

Stochastic Volatility (SV) model

- **Stochastic Volatility (SV) models** consist of an important class of asset price models in finance
- In a SV model, the asset price process is modeled as a SDE whose diffusion coefficient is also expressed as a SDE
  - In finance, diffusion coefficients are usually referred to as volatilities
- Thus, SV models are expressed as multi-dimensional SDEs (asset price process + volatility process) even for a single asset model

Heston model

- The **Heston model** is one of the most popular SV models in finance
- Let $\mu \in \mathbb{R}$, $\rho \in [-1, 1]$ and $\theta, v, \gamma > 0$ are constants
- Let $(B_{1,t})_{t\in[0,T]}$ and $(B_{2,t})_{t\in[0,T]}$ be two independent Brownian motions
- The Heston model is the SV model given by the solution of the following SDE:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_{1,t}, \\ dV_t = -\theta(V_t - v)dt + \gamma\sqrt{V_t}(\rho dB_{1,t} + \sqrt{1-\rho^2} dB_{2,t}). \end{cases} \tag{6}$$

Heston model

- In the Heston model, the process $(S_t)_{t\in[0,T]}$ usually represents a stock price process and $(V_t)_{t\in[0,T]}$ corresponds to the variance process of the return process of $(S_t)_{t\in[0,T]}$
- Setting $B_t := \rho B_{1,t} + \sqrt{1-\rho^2}B_{2,t}$, one can verify that the process $(B_t)_{t\in[0,T]}$ is a Brownian motion
- We can rewrite the SDE defining $(V_t)_{t\in[0,T]}$ as

$$dV_t = -\theta(V_t - v)dt + \gamma\sqrt{V_t}dB_t,$$

  which is the CIR model
- **R example**: heston.r