Unit 1: Mathematical Methods of Physics

Scope, Syllabus Map and Notation

This unit supplies every mathematical tool that Units 2–15 use. It is organised by the method a problem demands, not by the order in which the tools were historically invented. Each chapter ends with the traps that examiners set most often, and Chapter 13 collects the examination strategy and Chapters 14–15 the worked problems and formula sheet.

What the unit covers, and where.

Syllabus theme (GATE PH, CSIR-UGC NET Physical Sciences, Scientific Officer/Assistant)	Chapter
Vector algebra, scalar and vector products, triple products, index notation, orthogonal curvilinear coordinates	1
Limits, Taylor series, partial derivatives, Jacobians, Lagrange multipliers, integration techniques, Gamma and Beta functions	2
Gradient, divergence, curl, Laplacian; Gauss, Stokes and Green theorems; Dirac delta in vector calculus	3
Matrices, eigenvalues, diagonalisation, Cayley–Hamilton theorem, Hermitian and unitary operators, normal modes	4
Analytic functions, Cauchy theorems, Laurent series, residues, contour integrals	5
First- and second-order ODEs, Wronskian, Frobenius method, Sturm–Liouville theory	6
Legendre, Bessel, Hermite, Laguerre functions and spherical harmonics	7
Fourier series, Fourier and Laplace transforms, Dirac delta, convolution	8
Classification of PDEs, separation of variables (Laplace, heat, wave equations), Green's functions	9
Euler, Runge–Kutta, Newton–Raphson, trapezoidal and Simpson rules	10
Calculus of variations, tensors, group theory (SU(2), SO(3))	11
Mathematics inside physics: kinematics, circuits, heat flow, radiation	12
Examination toolkit, PYQ pattern map, worked problem bank (55 problems), formula sheet, 100-fact bank	13–16

Source of the syllabus map. The topic list follows the official benchmark summaries recorded in the Zenith forensic audit (GATE PH 2027 section on Mathematical Methods, and the Joint CSIR-UGC NET Physical Sciences Core and Advanced scheme), cross-checked against published syllabus listings. Error analysis, curve fitting and instrument-level numerical practice are treated in Unit 15; probability distributions used in thermodynamics are developed in Unit 6.

> **Notation used throughout.** Bold roman letters (A) are vectors; a hat ($\hat{n}$) marks a unit vector; $\hat{x},\hat{y},\hat{z}$ (or $\hat{i},\hat{j},\hat{k}$) are Cartesian unit vectors. δ_{ij} is the Kronecker delta, ε_{ijk} the Levi-Civita symbol, and repeated indices are summed (Einstein convention). Angles are in radians unless degrees are written. SI units are used in all numerical work.

Chapter 1: Vectors, Index Notation and Coordinate Systems

1.1 Vector algebra

A **scalar** has magnitude only (mass, temperature, energy). A **vector** has magnitude and direction and obeys the vector transformation law under rotation of axes. Displacement, velocity, force and electric field are vectors; a quantity that merely has a direction (an electric current through a wire, in the macroscopic sense) is not automatically a vector, because the test is the transformation law and the addition rule.

In Cartesian components, $\mathbf{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$, with magnitude $|A|=\sqrt{(A_x^2+A_y^2+A_z^2)}$ and unit vector $\hat{A}=\mathbf{A}/|A|$. Addition is component-wise; subtraction adds the negative vector, so every component of B changes sign:

$\mathbf{A}-\mathbf{B}=(A_x-B_x)\hat{x}+(A_y-B_y)\hat{y}+(A_z-B_z)\hat{z}$.

For two vectors of magnitudes A and B with angle θ between their directions when placed tail-to-tail, the resultant magnitude follows from the cosine rule:

$|\mathbf{A}+\mathbf{B}|=\sqrt{(A^2+B^2+2AB\cos\theta)}$, $|\mathbf{A}-\mathbf{B}|=\sqrt{(A^2+B^2-2AB\cos\theta)}$.

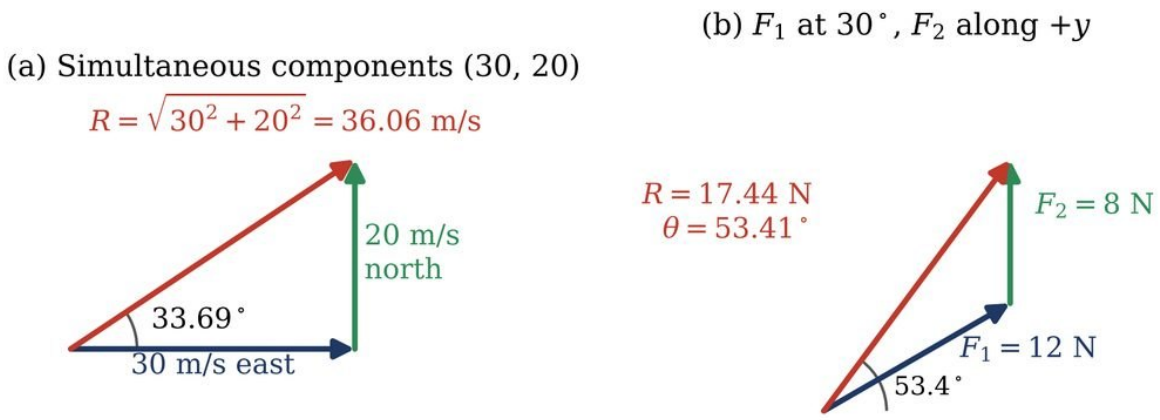


Figure 1.1: Resultants by component addition. The direction is always found as tan^(−1)(R_y/R_x) from the summed components.

Worked example 1.1 (two perpendicular velocities). A boat has 30 m/s east and 20 m/s north simultaneously. R=√(30²+20²)=√(1300)=36.06 m/s; θ=tan^(−1)(20/30)=33.69^∘ north of east.

Worked example 1.2 (force at an angle). F_1=12 N at 30^∘ to the x-axis and F□=8 N along +y. Components: R_x=12cos30^∘=10.392 N; R_y=12sin30^∘+8=14 N. So R=√(10.392²+14²)=√(304)=17.44 N at tan^(−1)(14/10.392)=53.41^∘ above the x-axis.

Worked example 1.3 (cosine rule). Forces of 6 N and 8 N act at 60^∘ to each other. R²=36+64+2(6)(8)cos60^∘=148, so R=12.17 N. (The resultant of two forces at 60^∘ always exceeds the larger force; 10 N would be the answer only at θ=90^∘.)

1.2 Scalar (dot) product

$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$.

It is commutative and distributive. It is the **projection** of one vector on the other times the length of the other (Figure 1.2a). $\mathbf{A}\cdot\mathbf{B}=0$ for non-zero vectors means perpendicular. Work $W=\mathbf{F}\cdot\mathbf{d}$, power $P=\mathbf{F}\cdot\mathbf{v}$ and flux $\Phi=\mathbf{E}\cdot\mathbf{A}$ are dot products.

1.3 Vector (cross) product

$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$,

where $\hat{\mathbf{n}}$ follows the right-hand rule. It is **anti-commutative** ($\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$), not associative, and $|\mathbf{A}\times\mathbf{B}|$ equals the area of the parallelogram on A,B. Torque $\tau=\mathbf{r}\times\mathbf{F}$, angular momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$, Lorentz force $\mathbf{F}=q\mathbf{v}\times\mathbf{B}$ and the Poynting vector are cross products. Strictly, $\mathbf{A}\times\mathbf{B}$ is a pseudovector (axial vector): under inversion $\mathbf{r}\to-\mathbf{r}$ it does not change sign.

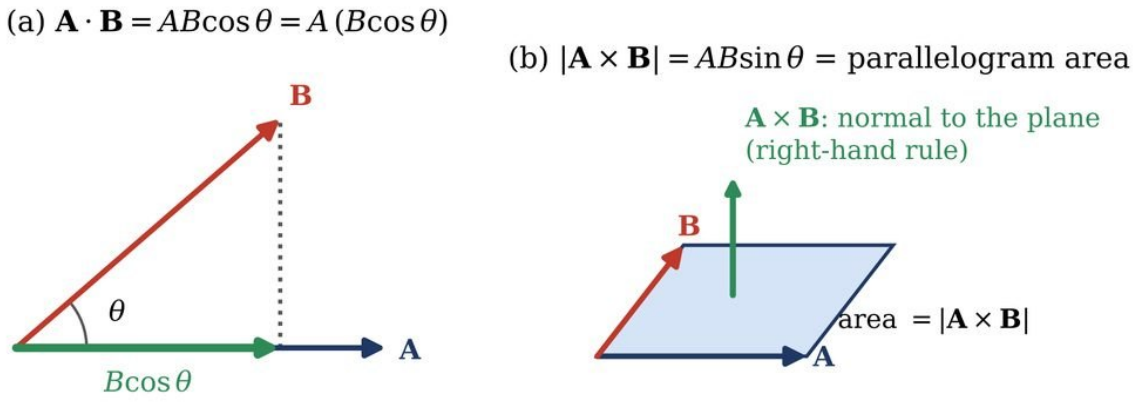


***Figure 1.2**: (a) Dot product as projection; (b) cross product magnitude as parallelogram area and its direction normal to the plane.*

Worked example 1.4. $\mathbf{A}=2\hat{\mathbf{x}}+3\hat{\mathbf{y}}$, $\mathbf{B}=-\hat{\mathbf{x}}+4\hat{\mathbf{y}}$. Then $\mathbf{A}-\mathbf{B}=3\hat{\mathbf{x}}-\hat{\mathbf{y}}$; $\mathbf{A}\cdot\mathbf{B}=-2+12=10$; $\mathbf{A}\times\mathbf{B}=(2\cdot4-3\cdot(-1))\hat{\mathbf{z}}=11\hat{\mathbf{z}}$. Check: $|\mathbf{A}||\mathbf{B}|\sin\theta=\sqrt{13}\sqrt{17}\sin\theta=11$ gives $\sin\theta=0.7399$, and $\cos\theta=10/\sqrt{221}=0.6727$; $0.7399^2+0.6727^2=1.000$ ✓.

1.4 Triple products and identities

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$.

Its modulus is the volume of the parallelepiped on the three vectors; it vanishes when the vectors are coplanar. The **BAC–CAB** rule for the vector triple product is

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.

Two further identities used constantly:

$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$, $\quad |\mathbf{A}\times\mathbf{B}|^2 = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2$.

Trap. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\neq(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$; the second equals $\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$. The position of the brackets decides which vector is multiplied by which dot product.

1.5 Index notation: Kronecker delta and Levi-Civita symbol

Index notation turns vector identities into algebra. With $\delta_{ij}=1$ for $i=j$ and 0 otherwise, and $\varepsilon_{ijk}=+1$ for cyclic permutations of 123, -1 for anti-cyclic ones, and 0 if any two indices coincide:

$$\mathbf{A}\cdot\mathbf{B}=A_iB_i, \qquad (\mathbf{A}\times\mathbf{B})_i=\varepsilon_{ijk}A_jB_k, \qquad \delta_{ii}=3, \qquad \delta_{ij}A_j=A_i,$$

$$\varepsilon_{ijk}\varepsilon_{ilm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}, \qquad \varepsilon_{ijk}\varepsilon_{ijm}=2\delta_{km}, \qquad \varepsilon_{ijk}\varepsilon_{ijk}=6.$$

The first contraction identity proves BAC–CAB in two lines: $[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_i=\varepsilon_{ijk}A_j\varepsilon_{klm}B_lC_m=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})A_jB_lC_m=B_i(\mathbf{A}\cdot\mathbf{C})-C_i(\mathbf{A}\cdot\mathbf{B})$.

1.6 Orthogonal curvilinear coordinates

Let (q_1,q_2,q_3) be orthogonal coordinates with position $\mathbf{r}(q_1,q_2,q_3)$. The **scale factors** are $h_i=\left|\partial\mathbf{r}/\partial q_i\right|$; the line element is $d\ell=\sum_i h_i\,dq_i\,\hat{\mathbf{e}}_i$ and the volume element is $dV=h_1h_2h_3\,dq_1dq_2dq_3$.

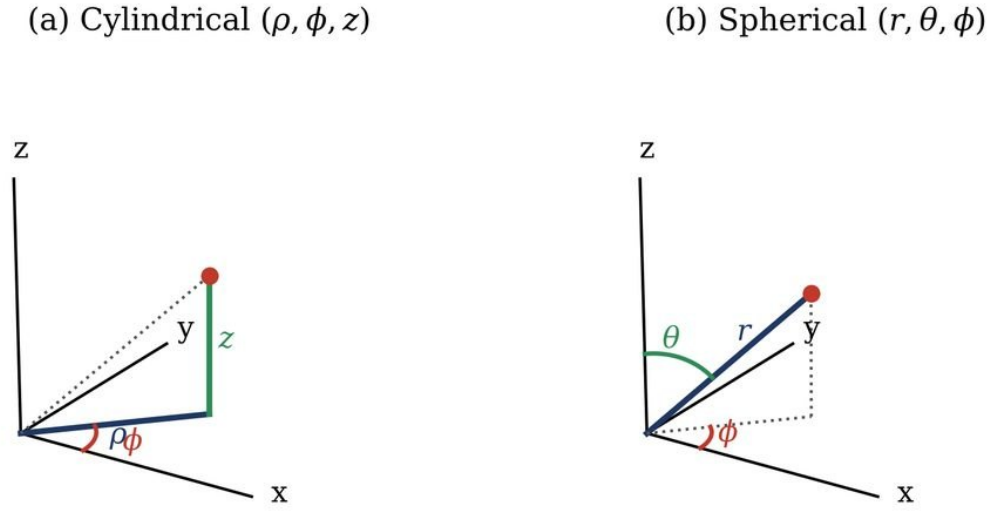


Figure 1.3: Cylindrical and spherical coordinates. The unit vectors $\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\phi}},\hat{\mathbf{r}},\hat{\boldsymbol{\theta}}$ point in the direction of increasing coordinate and change with position.

System	Relations to Cartesian	(h_1,h_2,h_3)	dV
Cartesian (x,y,z)	—	$(1,1,1)$	$dx\,dy\,dz$
Cylindrical (ρ,ϕ,z)	$x=\rho\cos\phi$, $y=\rho\sin\phi$	$(1,\rho,1)$	$\rho\,d\rho\,d\phi\,dz$
Spherical (r,θ,ϕ)	$x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$	$(1,r,r\sin\theta)$	$r^2\sin\theta\,dr\,d\theta\,d\phi$

The unit vectors of curvilinear systems are position-dependent, so $\partial\hat{\boldsymbol{\rho}}/\partial\phi=\hat{\boldsymbol{\phi}}$ and $\partial\hat{\boldsymbol{\phi}}/\partial\phi=-\hat{\boldsymbol{\rho}}$ in cylindrical coordinates. Velocity and acceleration in plane polar form follow:

$$\mathbf{v}=\dot r\,\hat{\mathbf{r}}+r\dot\theta\,\hat{\boldsymbol{\theta}}, \qquad \mathbf{a}=(\ddot r-r\dot\theta^2)\hat{\mathbf{r}}+(r\ddot\theta+2\dot r\dot\theta)\hat{\boldsymbol{\theta}}.$$

The term $-r\dot\theta^2$ is the centripetal acceleration and $2\dot r\dot\theta$ is the Coriolis term.

1.7 Rotation of axes and the transformation law

If the axes are rotated, the components of a vector change but the vector does not. With direction cosines $R_{ij}=\hat{\mathbf{e}}'_i\cdot\hat{\mathbf{e}}_j$, the new components are $A'_i=R_{ij}A_j$. The matrix R is **orthogonal**, $R^{T}R=I$, so $\det R=\pm1$: $+1$ for a proper rotation, -1 for a reflection or inversion. The direction cosines of a line obey $l^2+m^2+n^2=1$. For a rotation of the axes by θ about z,

$$R_z(\theta)=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Invariants such as $\mathbf{A}\cdot\mathbf{B}$ and $|\mathbf{A}|$ survive any orthogonal transformation, and this is the definition of a scalar: the dot product of two vectors is a scalar because $R^{T}R=I$. A pseudovector acquires an extra factor $\det R$ under R.

> **Worked example 1.5.** Rotate the axes by 90° about z. The old unit vector $\hat{\mathbf{x}}=(1,0,0)$ has new components $R_z(90^\circ)(1,0,0)^{T}=(0,-1,0)$: it now points along $-\hat{\mathbf{y}}'$. Passive and active rotations differ by the sign of θ.

1.8 Vectors that change in time; rotating frames

For a vector of constant magnitude rotating with angular velocity $\boldsymbol{\omega}$, $\dot{\mathbf{A}}=\boldsymbol{\omega}\times\mathbf{A}$. In a frame rotating with constant $\boldsymbol{\Omega}$ relative to an inertial frame,

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{in}}=\left(\frac{d\mathbf{A}}{dt}\right)_{\text{rot}}+\boldsymbol{\Omega}\times\mathbf{A},$$

$$\mathbf{v}_{\text{in}}=\mathbf{v}_{\text{rot}}+\boldsymbol{\Omega}\times\mathbf{r}, \qquad \mathbf{a}_{\text{rot}}=\mathbf{a}_{\text{in}}-2\boldsymbol{\Omega}\times\mathbf{v}_{\text{rot}}-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}).$$

The two extra terms define the **Coriolis** force $-2m\boldsymbol{\Omega}\times\mathbf{v}_{\text{rot}}$ (perpendicular to the velocity, does no work) and the **centrifugal** force $-m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})=m\Omega^2\mathbf{r}_\perp$ (directed away from the axis). They are fictitious in the sense that they arise only from the choice of frame (Unit 2 develops them).

> **Worked example 1.6 (Coriolis deflection).** An object dropped from rest from $h=100$ m at latitude 45° falls for $t=\sqrt{2h/g}=4.52$ s. The Coriolis acceleration

deflects it eastward by $d = \tfrac{1}{3}\Omega g t^3 \cos\lambda$ with $\Omega = 7.292\times10^{-5}$ rad/s: $d = \tfrac{1}{3}(7.292\times10^{-5})(9.8)(92.2)(0.7071) \approx 1.55\times10^{-2}$ m, i.e. about 1.6 cm.

1.9 Dimensional analysis

Every physical equation must be **dimensionally homogeneous**: all terms have the same dimensions. Only quantities of equal dimension can be added, and arguments of exponentials, logarithms and trigonometric functions must be dimensionless. The base dimensions are [M],[L],[T],[I],[Θ],[N],[J] (mass, length, time, current, temperature, amount, luminous intensity).

Quantity	Dimensions	SI unit
Force	MLT^{-2}	N
Energy, work, torque	ML^2T^{-2}	J, N m
Power	ML^2T^{-3}	W
Pressure	$ML^{-1}T^{-2}$	Pa
Planck constant h, angular momentum	ML^2T^{-1}	J s
Dynamic viscosity η	$ML^{-1}T^{-1}$	Pa s
Kinematic viscosity, diffusivity	L^2T^{-1}	m^2/s
Electric field	$MLT^{-3}I^{-1}$	V/m
Magnetic flux density B	$MT^{-2}I^{-1}$	T
Entropy, Boltzmann constant k_B	$ML^2T^{-2}\Theta^{-1}$	J/K
Gravitational constant G	$M^{-1}L^3T^{-2}$	$N\,m^2/kg^2$
Permittivity ε_0	$M^{-1}L^{-3}T^4I^2$	F/m

Buckingham π theorem. A physical relation among n variables that involve k independent base dimensions can be written as a relation among $n-k$ dimensionless groups. Dimensional analysis fixes exponents but not dimensionless constants.

Worked example 1.7 (pendulum). Suppose T depends on ℓ, m and g: $T = C\ell^a m^b g^c$. Matching $[T]=T^1$ gives b=0 (no mass), a+c=0 and −2c=1, so c=−1/2, a=1/2: $T = C\sqrt{\ell/g}$, with $C = 2\pi$ from the dynamics.

Worked example 1.8 (blast wave). The radius R of a strong explosion depends on energy E, air density ρ and time t. $[E]=ML^2T^{-2}$, $[\rho]=ML^{-3}$. $R = CE^a\rho^b t^c$: mass requires a+b=0; length 2a−3b=1; time −2a+c=0. So a=1/5, b=−1/5, c=2/5, giving $R = C(Et^2/\rho)^{1/5}$, the Taylor–Sedov scaling.

Traps. (1) A dimensionally consistent equation can still be wrong (a missing 1/2 or 2π). (2) Radians and steradians are dimensionless; degrees are not "dimensions" but units. (3) Dimensional analysis cannot distinguish a polar vector from a pseudovector; $r\times F$ and $r\cdot F$ both have the dimensions of energy. (4) A rotation about the wrong axis, or passive vs active, flips the sign of θ in R_z.

1.10 Chapter problems with solutions

1. **Show** $A\cdot(B\times C) = -B\cdot(A\times C)$**.** Swapping two rows of the determinant changes its sign.
2. **For** A=(1,2,3)**,** B=(2,−1,1)**,** C=(0,1,4) **compute** $A\cdot B$**,** $A\times B$**,** $A\cdot(B\times C)$**.** $A\cdot B = 2-2+3 = 3$; $A\times B = (2\cdot1-3(-1),\ 3\cdot2-1\cdot1,\ 1(-1)-2\cdot2) = (5,5,-5)$; $B\times C = (-1\cdot4-1\cdot1,\ 1\cdot0-2\cdot4,\ 2\cdot1-0) = (-5,-8,2)$, and $A\cdot(B\times C) = -5-16+6 = -15$. Check with BAC–CAB: $A\times(B\times C) = B(A\cdot C) - C(A\cdot B) = 14B - 3C = (28,-17,2)$ ✓.
3. **Angle between** A **and** B **above.** $\cos\theta = 3/(\sqrt{14}\sqrt{6}) = 0.3273$, $\theta = 70.9^\circ$.
4. **Area of the triangle with vertices** (1,0,0),(0,2,0),(0,0,3)**.** $\tfrac{1}{2}|(-1,2,0)\times(-1,0,3)| = \tfrac{1}{2}|(6,3,2)| = \tfrac{1}{2}\cdot7 = 3.5$.
5. **A force** $F = 5\hat{z}$ **N acts at** r=(2,3,0) **m. Find the torque about the origin.** $\tau = r\times F = (15,-10,0)$ N m, magnitude 18.03 N m.
6. **Prove** $A\times B$ **is perpendicular to** A**.** $A\cdot(A\times B) = \varepsilon_{ijk}A_iA_jB_k = 0$ because ε_{ijk} is antisymmetric in ij while A_iA_j is symmetric.
7. **Find the volume element for** $\rho\in[0,R]$**,** $z\in[0,h]$**,** $\phi\in[0,2\pi)$**.** $V = \int \rho\, d\rho\, d\phi\, dz = \pi R^2 h$ ✓.
8. **Show that the Coriolis force does no work.** Its power is $-2m(\Omega\times v)\cdot v = 0$ since the cross product is perpendicular to v.
9. **Find a dimensionless group built from viscosity** η**, density** ρ**, speed** v **and length** L**.** $\mathrm{Re} = \rho vL/\eta$ (Reynolds number): ρvL has $ML^{-1}T^{-1}$, same as η.
10. **Express the unit vectors** $\hat{\rho},\hat{\phi}$ **in Cartesian components.** $\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$, $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$; they are orthonormal and $\hat{\rho}\times\hat{\phi} = \hat{z}$.

One-look revision (Chapter 1). Dot → scalar, projection, commutative. Cross → pseudovector, area, anti-commutative. Triple scalar product = signed volume = determinant. BAC–CAB: $A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$. $\varepsilon\varepsilon = \delta\delta - \delta\delta$. Spherical $dV = r^2\sin\theta\, dr\, d\theta\, d\phi$; cylindrical $dV = \rho\, d\rho\, d\phi\, dz$. Direction of a resultant always comes from summed components, never from a "hypotenuse by eye".

Chapter 2: Calculus Toolkit for Physics

2.1 Limits, continuity and the derivative

lim_(x→a)f(x)=L means that for every ε>0 there is δ>0 with |f(x)−L|<ε whenever 0<|x−a|<δ. The derivative is f′(a)=lim_(h→0)[f(a+h)−f(a)]/h. Differentiability implies continuity; the converse fails (|x| at 0).

Rules: product (fg)′=f′g+fg′; quotient (f/g)′=(f′g−fg′)/g²; chain d/(dx)f(g(x))=f′(g) g′. **Mean value theorem:** if f is continuous on [a,b] and differentiable on (a,b), some c□(a,b) has f′(c)=[f(b)−f(a)]/(b−a). **L'Hôpital:** for 0/0 or ∞/∞ forms, limf/g=limf′/g′ if the latter exists.

Standard limits: lim_(x→0)(sinx)/x=1, lim_(x→0)(1−cosx)/(x²)=1/2, lim_(x→0)(1+x)^(1/x)=e, lim_(n→∞)(1+a/n)ⁿ=e^a.

> **Worked example 2.1.** lim_(x→0)(e^x−1−x)/(x²) is 0/0. Differentiating twice (or using e^x=1+x+x²/2+…) gives 1/2.

2.2 Taylor and Maclaurin series

f(x)=∑_(n=0)^∞ (f⁽ⁿ⁾(a))/(n!)(x−a)ⁿ, R_N=(f^(N+1)(ξ))/((N+1)!)(x−a)^(N+1) Lagrange remainder.

Function	Series about 0	Valid for
e^x	1+x+(x²)/(2!)+(x³)/(3!)+⋯	all x
sinx	x−(x³)/(3!)+(x⋯)/(5!)−□	all x
cosx	1−(x²)/(2!)+(x⁴)/(4!)−⋯	all x
ln(1+x)	x−(x²)/2+(x³)/3−⋯	−1<x≤1
(1+x)ⁿ	1+nx+(n(n−1))/(2!)x²+⋯	|x|<1
1/(1−x)	1+x+x²+⋯	|x|<1
tanx	x+(x³)/3+(2x⁵)/(15)+⋯	|x|<π/2
tanhx	x−(x³)/3+(2x⁵)/(15)−⋯	|x|<π/2

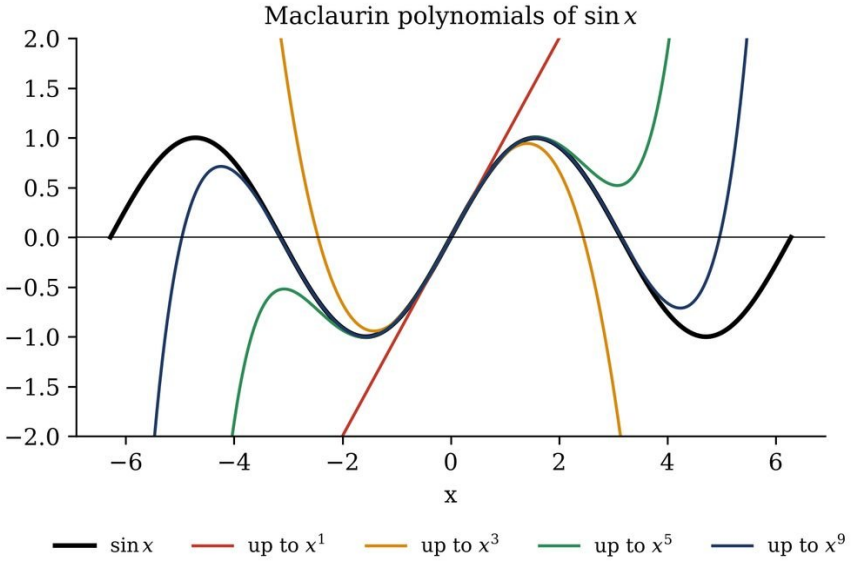


Figure 2.1: Maclaurin polynomials converge to sinx over a widening interval; each polynomial is accurate only near the expansion point.

Physical uses: small-angle pendulum (sinθ≈θ), relativistic energy E≈mc²+1/2mv² for v≪c from γ=(1−v²/c²)^(−1/2)≈1+(v²)/(2c²), and the binomial expansion of potentials in multipole expansions.

2.3 Partial derivatives, total differential and chain rule

For f(x,y), ∂f/∂x holds y fixed. The total differential is df=(∂f)/(∂x)dx+(∂f)/(∂y)dy. For f(x(t),y(t)): (df)/(dt)=(∂f)/(∂x)ẋ+(∂f)/(∂y)ẏ. Mixed partials commute, f_(xy)=f_(yx), when they are continuous.

Exact differential. M dx+N dy is exact iff ∂M/∂y=∂N/∂x. Thermodynamic state functions have exact differentials (dU, dS), whereas δQ and δW do not.

Implicit and cyclic relations: if F(x,y,z)=0 then ((∂x)/(∂y))_z((∂y)/(∂z))_x((∂z)/(∂x))_y=−1.

(The minus sign is the usual error: the reciprocal rule (∂x/∂y)_z=1/(∂y/∂x)_z is positive, the cyclic rule is −1.)

Stationary points. At ∇f=0, with D=f_(xx)f_(yy)−f_(xy)²: D>0,f_(xx)>0 → minimum; D>0,f_(xx)<0 → maximum; D<0 → saddle; D=0 → test inconclusive.

2.4 Jacobians and change of variables

For (x,y)→(u,v), dx dy=|J| du dv with J=∂(x,y)/∂(u,v)=x_uy_v−x_vy_u. For polar coordinates J=r; for spherical coordinates J=r²sinθ (Section 1.6).

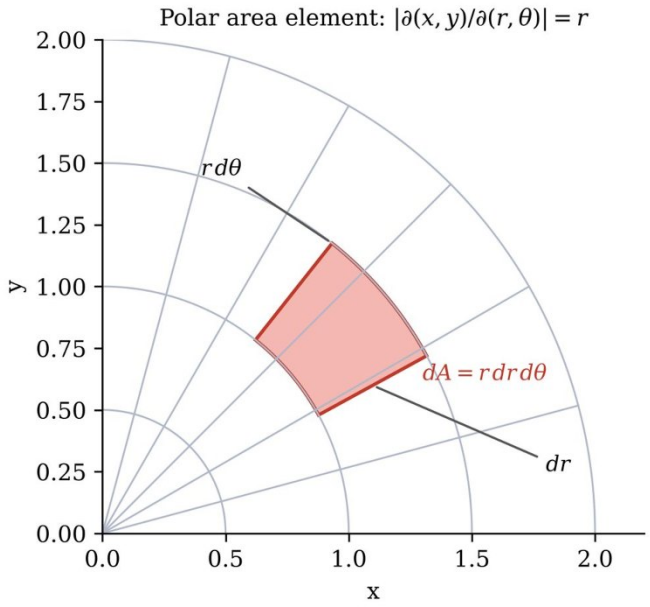


Figure 2.2: The Jacobian of the polar map is r: the shaded patch has sides dr and r dθ.

Worked example 2.2 (Gaussian integral). I=∫_(−∞)^∞ e^(−x²)dx. Then I²=∫∫ e^(−(x²+y²))dx dy=∫_0^(2π) ∫_0^∞ e^(−r²)r dr dθ=2π · 1/2=π, so I=√π and ∫_(−∞)^∞ e^(−ax²)dx=√(π/a).

2.5 Constrained extrema: Lagrange multipliers

To extremise f(x) subject to g_k(x)=0, k=1…m, at a regular point (the ∇g_k linearly independent) the condition is

∇f=∑_(k=1)^m λ_k∇g_k, g_k=0,

with **one multiplier per constraint**. (Writing a single λ for several constraints is wrong.)

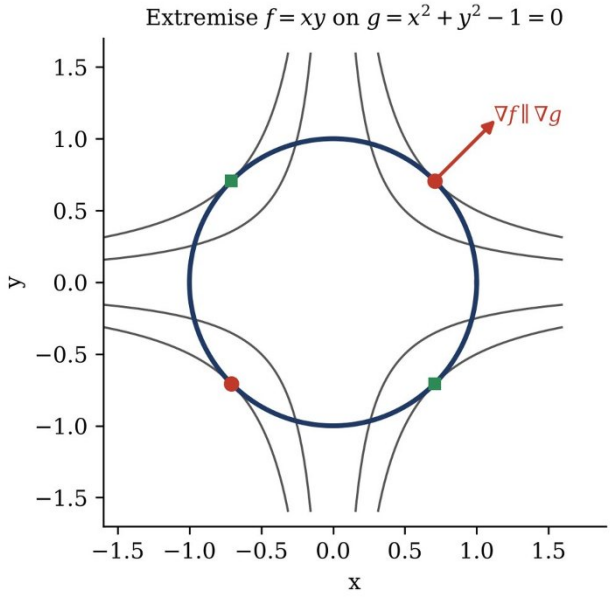


Figure 2.3: Extrema of f=xy on the unit circle occur where level curves of f touch the constraint, so ∇f∥∇g. Circles mark maxima (f=1/2), squares minima (f=−1/2).

Worked example 2.3. Extremise f=xy on x²+y²=1. y=2λx, x=2λy ⇒ x²=y², so x=±y=±1/√2. Maximum f=1/2 where x,y have the same sign; minimum f=−1/2 where they differ.

Worked example 2.4 (two constraints). Extremise f=z on the curve g₁=x²+y²−1=0, g□=x+y+z−1=0. ∇f=(0,0,1)=λ₁(2x,2y,0)+λ₂(1,1,1) gives λ₂=1 and 2λ₁x=2λ₁y=−1, so x=y=±1/√2 and z=1−x−y. Maximum z=1+√2 at x=y=−1/√2; minimum z=1−√2 at x=y=+1/√2.

2.6 Integration techniques

- **Substitution:** ∫f(g(x))g′(x) dx=∫f(u) du.
- **By parts:** ∫u dv=uv−∫v du (choose u by LIATE: Logarithmic, Inverse trig, Algebraic, Trigonometric, Exponential).
- **Leibniz rule (differentiation under the integral):** d/(dx)∫_(a(x))^(b(x)) f(x,t) dt=f(x,b)b′−f(x,a)a′+∫_a^b (∂f)/(∂x)dt.
- **Symmetry:** over symmetric limits an odd integrand integrates to zero; ∫_0^a f(x)dx=∫_0^a f(a−x)dx.

- **Improper integrals** converge if the limit exists; $\int_1^\infty x^{-p}dx$ converges iff $p>1$; $\int_0^1 x^{-p}dx$ converges iff $p<1$.

Worked example 2.5. $\int_0^\infty xe^{-ax}dx$: by parts with $u=x$, $dv=e^{-ax}dx$ gives $1/a^2$. In general $\int_0^\infty x^n e^{-ax}dx=n!/a^{n+1}$ for integer $n\geq 0$.

2.7 Gamma and Beta functions

$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt$ (Re $z>0$), $\Gamma(z+1)=z\,\Gamma(z)$, $\Gamma(n+1)=n!$, $\Gamma(1/2)=\sqrt{\pi}$,
$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}dt=(\Gamma(m)\Gamma(n))/(\Gamma(m+n))$, $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\tfrac{1}{2}B((p+1)/2,(q+1)/2)$.
Reflection: $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$. Stirling: $\ln n!\approx n\ln n-n$ (used in Unit 6 for entropy). $\Gamma(z)$ has simple poles at $z=0,-1,-2,\ldots$

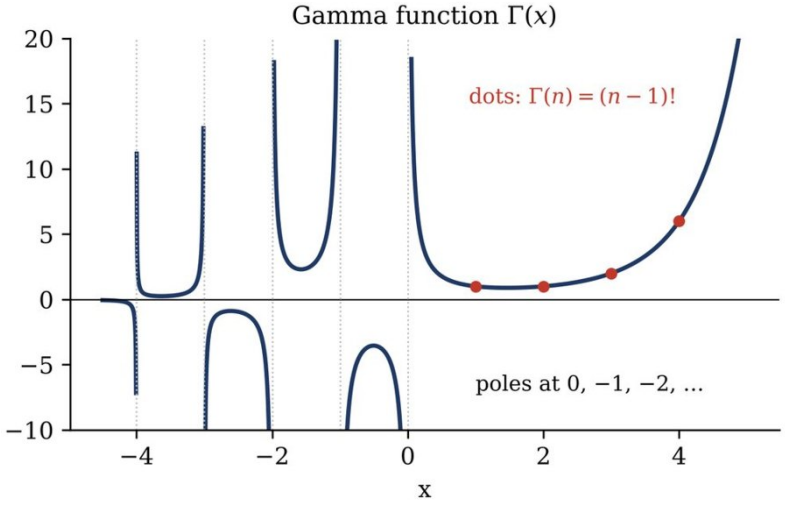


Figure 2.4: $\Gamma(x)$ interpolates the factorial and has simple poles at non-positive integers.

Worked example 2.6. $\Gamma(5/2)=3/2\cdot 1/2\,\Gamma(1/2)=3/4\sqrt{\pi}=1.3293$. Also $\int_0^{\pi/2}\sin^4\theta\,d\theta=1/2B(5/2,1/2)=1/2(\Gamma(5/2)\Gamma(1/2))/(\Gamma(3))=(3\pi)/(16)$.

2.8 Infinite series and convergence

A series $\sum a_n$ converges if its partial sums approach a finite limit. Necessary (not sufficient) condition: $a_n\to 0$.

Test	Statement
Ratio (d'Alembert)	$L=\lim\lvert a_{n+1}/a_n\rvert$: converges absolutely if $L<1$, diverges if $L>1$, inconclusive if $L=1$
Root (Cauchy)	$L=\lim\lvert a_n\rvert^{1/n}$ with the same verdicts
Comparison	$0\leq a_n\leq b_n$ and $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges
Integral	$\sum f(n)$ and $\int^\infty f\,dx$ converge together, for positive decreasing f
Alternating (Leibniz)	a_n decreasing to 0 $\Rightarrow$ $\sum(-1)^n a_n$ converges
p-series	$\sum n^{-p}$ converges iff $p>1$; the harmonic series $p=1$ diverges

A **power series** $\sum c_n(x-a)^n$ converges for $\lvert x-a\rvert<R$ with $R=\lim\lvert c_n/c_{n+1}\rvert$; the end points must be tested separately. Inside the radius of convergence, a power series may be differentiated and integrated term by term. **Absolute convergence** permits rearrangement; conditional convergence does not (Riemann).

Worked example 2.7. $\sum x^n/n^2$ has $R=1$; at $x=\pm 1$ it converges absolutely ($p=2$). By contrast $\sum x^n/n$ has $R=1$, converges at $x=-1$ (alternating) and diverges at $x=+1$ (harmonic).

2.9 Multiple integrals in physics

Volumes and moments are computed in the coordinate system that matches the symmetry; the Jacobian supplies the measure.

Worked example 2.8 (volume of a sphere). $V=\int_0^R\int_0^\pi\int_0^{2\pi} r^2\sin\theta\,d\phi\,d\theta\,dr=(R^3)/3\cdot 2\cdot 2\pi=(4\pi R^3)/3$.
Worked example 2.9 (moment of inertia of a disc). Uniform disc, mass M, radius R, surface density $\sigma=M/\pi R^2$, axis through the centre and perpendicular to the plane: $I=\int \sigma r^2\,r\,dr\,d\phi=\sigma\cdot 2\pi(R^4)/4=1/2MR^2$.
Worked example 2.10 (centre of mass). Solid uniform hemisphere of radius R: $z_{cm}=(\int z\,dV)/(\int dV)=(\int_0^R\int_0^{\pi/2}\int_0^{2\pi} r\cos\theta\, r^2\sin\theta\,d\phi\,d\theta\,dr)/(2/3\pi R^3)=(2\pi\cdot(R^4)/4\cdot 1/2)/(2/3\pi R^3)=(3R)/8$.

2.10 Differentiation under the integral sign

The Leibniz rule (Section 2.6) evaluates integrals that resist direct methods.

Worked example 2.11. Define $I(a)=\int_0^\infty (\sin x)/x e^{-ax}dx$. Then $I'(a)=-\int_0^\infty \sin x\, e^{-ax}dx=-1/(1+a^2)$, and $I(\infty)=0$ gives $I(a)=\pi/2-\tan^{-1}a$. Setting $a=0$ yields the Dirichlet integral $\int_0^\infty (\sin x)/x dx=\pi/2$.

2.11 Error function, Stirling's formula and elliptic integrals

The **error function** erf(x)=2/(√π)∫₀^x e^(−t²)dt satisfies erf(∞)=1, erf(−x)=−erf(x), and erfc=1−erf; it arises in diffusion and in Gaussian probabilities, where P(|x−μ|<σ)=erf(1/√2)=0.683. A refined **Stirling formula** is n!≈√(2πn) (n/e)ⁿ, so lnn!=nlnn−n+1/2ln(2πn)+… (the last term is negligible for the n∼10²³ of statistical physics).

The exact period of a simple pendulum of amplitude θ_0 is T=4√(ℓ/g) K(k), k=sin(θ_0/2), where K is the complete elliptic integral of the first kind; T≈T_0(1+(θ_0^2)/(16)+⋯) with T_0=2π√(ℓ/g). For θ_0=60^∘ the exact ratio is T/T_0=2K(1/2)/π=1.0732, against 1.0685 from the first-order correction.

2.12 Hyperbolic functions

coshx=1/2(e^x+e^(−x)), sinhx=1/2(e^x−e^(−x)), cosh²x−sinh²x=1, d/(dx)sinhx=coshx, d/(dx)coshx=sinhx, coshix=cosx, sinhix=isinx. They appear in catenaries, relativistic rapidity (v=ctanhη) and the solutions of y″=k²y with boundary conditions.

2.13 Chapter problems with solutions

1. lim_(x→0)(tanx−x)/x³**.** Using tanx=x+x³/3+⋯: the limit is 1/3.
2. **Does** ∑n!/nⁿ **converge?** Ratio: ((n+1)!/(n+1)^(n+1))/(n!/nⁿ)=(n/(n+1))ⁿ→e^(−1)<1; it converges.
3. ∫₀^∞ xe^(−x²)dx**.** 1/2 (substitute u=x²).
4. **Maximise** f=x²y **subject to** x+y=3**,** x,y>0**.** Lagrange: 2xy=λ, x²=λ ⇒ x=2y, so x=2, y=1, f_(max)=4.
5. **Stationary points of** f=x³−3xy+y³**.** ∇f=0 gives y=x² and x=y² ⇒ (0,0) and (1,1). At (1,1): f_(xx)=6,f_(yy)=6,f_(xy)=−3, D=27>0 ⇒ minimum (f=−1). At (0,0): D=−9<0 ⇒ saddle.
6. **Is** (2xy+3)dx+(x²−1)dy **exact?** M_y=2x=N_x ⇒ yes; the potential is F=x²y+3x−y.
7. Γ(−1/2)**.** Γ(1/2)=(−1/2)Γ(−1/2) ⇒ Γ(−1/2)=−2√π=−3.545.
8. ∫₀¹ x⁴(1−x)²dx**.** B(5,3)=(4! 2!)/(7!)=1/(105).
9. **Radius of convergence of** ∑n!xⁿ**.** |c_n/c_(n+1)|=1/(n+1)→0; R=0 (convergent only at x=0).
10. **Show** ∫_(−∞)^∞ x²e^(−x²/2σ²)dx=σ³√(2π)**.** Differentiate ∫e^(−ax²)dx=√(π/a) with respect to a: ∫x²e^(−ax²)dx=(√π)/2a^(−3/2); with a=1/2σ² this equals σ³√(2π), so the Gaussian variance is σ².

Traps. (1) Γ(n)=(n−1)!, not n!. (2) lim_(x→0)sinx/x=1 only in radians. (3) The Taylor series of ln(1+x) converges for x=1 but not x=−1. (4) Taking the exam "derivative" of x^x as x · x^(x−1) is wrong: d/(dx)x^x=x^x(1+lnx).

One-look revision (Chapter 2). MVT; L'Hôpital; Maclaurin table; ∂M/∂y=∂N/∂x for exactness; cyclic rule =−1; Jacobian polar =r; Lagrange ∇f=∑λ_k∇g_k; ∫e^(−ax²)=√(π/a); Γ(1/2)=√π; B(m,n)=Γ(m)Γ(n)/Γ(m+n).

Chapter 3: Vector Calculus

3.1 Gradient, divergence, curl and Laplacian

With ∇=x̂∂_x+ŷ∂_y+ẑ∂_z:

∇f=((∂f)/(∂x),(∂f)/(∂y),(∂f)/(∂z)), ∇ · F=(∂F_x)/(∂x)+(∂F_y)/(∂y)+(∂F_z)/(∂z), ∇×F=|[x̂, ŷ, ẑ; _x, _y, _z; F_x, F_y, F_z]|, ∂ ²f=∂∂f.∇∇ · ∇

- **Gradient** points in the direction of steepest increase of f; its magnitude is the maximal rate of change; it is perpendicular to the level surfaces f=const. The directional derivative along unit û is û · ∇f. Force from potential: F=−∇V; electric field E=−∇Φ.
- **Divergence** measures net outflow per unit volume (source density).
- **Curl** measures local circulation; (∇×F) · n̂ is the circulation per unit area about n̂.

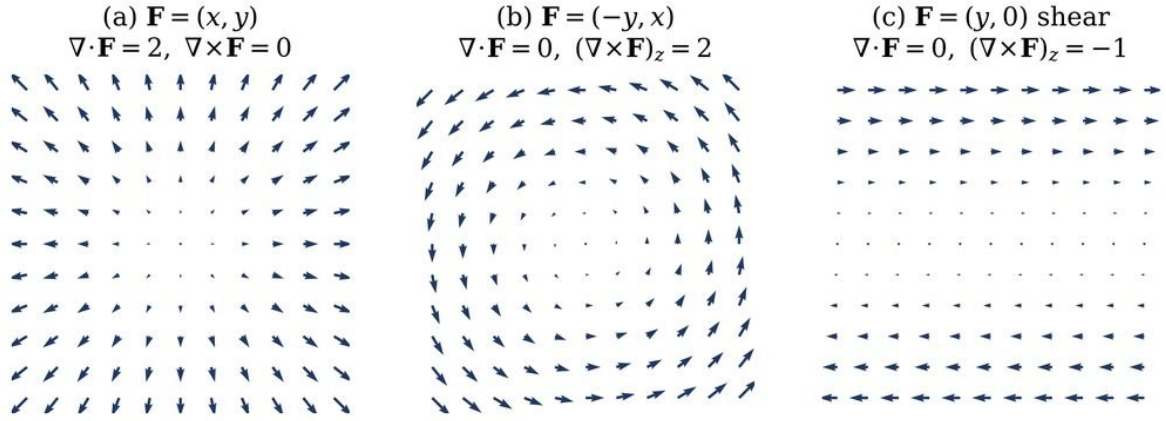


Figure 3.1: (a) A radial field has non-zero divergence and zero curl; (b) a rigid rotation has zero divergence and non-zero curl; (c) a shear flow has zero divergence yet non-zero curl, so "curl means swirling paths" is not literally true.

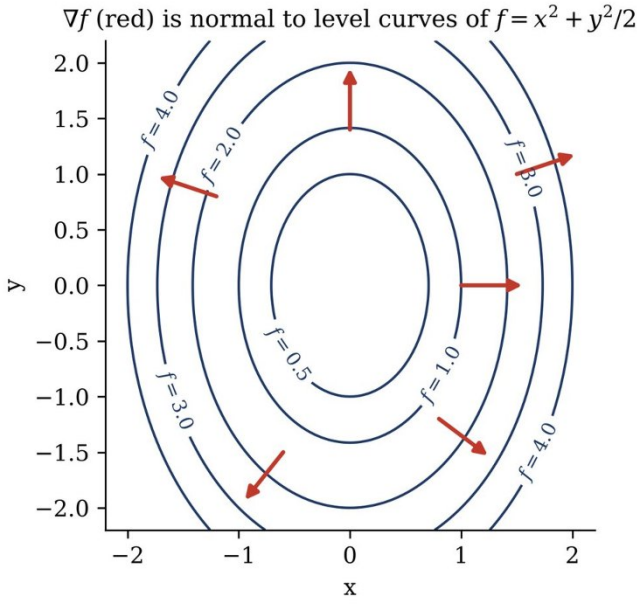


Figure 3.2: Level curves of f=x²+y²/2; ∇f is normal to each level curve and points toward increasing f.

3.2 Identities

$\nabla\times(\nabla f)=0$, $\nabla\cdot(\nabla\times F)=0$, $\nabla\times(\nabla\times F)=\nabla(\nabla\cdot F)-\nabla^2 F$,
$\nabla(fg)=f\nabla g+g\nabla f$, $\nabla\cdot(fF)=f\nabla\cdot F+F\cdot\nabla f$, $\nabla\times(fF)=f\nabla\times F+\nabla f\times F$,
$\nabla\cdot(A\times B)=B\cdot(\nabla\times A)-A\cdot(\nabla\times B)$.
Radial results with r=(x,y,z), r=|r|: $\nabla\hat{r}=\hat{r}$, $\nabla r^n=nr^{n-1}\hat{r}$, $\nabla\cdot r=3$, $\nabla\times r=0$, $\nabla\cdot(r^n r)=(n+3)r^n$, $\nabla^2 r^n=n(n+1)r^{n-2}$, and $\nabla^2(1/r)=0$ for $r\neq 0$. For a central function $\nabla\times[f(r)r]=0$.

3.3 Operators in curvilinear coordinates

With scale factors h_i (Section 1.6):
$\nabla f=\sum_i (\hat{e}_i)/(h_i)(\partial f)/(\partial q_i)$, $\nabla\cdot F=1/(h_1h_2h_3)\sum_{cyc}\partial/(\partial q_1)(h_2h_3F_1)$, $\nabla^2 f=1/(h_1h_2h_3)\sum_{cyc}\partial/(\partial q_1)((h_2h_3)/(h_1)(\partial f)/(\partial q_1))$.
The two Laplacians you must know by heart:
$\nabla^2 f=1/\rho\partial/(\partial\rho)(\rho(\partial f)/(\partial\rho))+1/(\rho^2)(\partial^2 f)/(\partial\phi^2)+(\partial^2 f)/(\partial z^2)$ (cylindrical),
$\nabla^2 f=1/(r^2)\partial/(\partial r)(r^2(\partial f)/(\partial r))+1/(r^2\sin\theta)\partial/(\partial\theta)(\sin\theta(\partial f)/(\partial\theta))+1/(r^2\sin^2\theta)(\partial^2 f)/(\partial\phi^2)$ (spherical).
Spherical divergence: $\nabla\cdot F=1/(r^2)\partial_r(r^2F_r)+1/(r\sin\theta)\partial_\theta(\sin\theta F_\theta)+1/(r\sin\theta)\partial_\phi F_\phi$.

3.4 Line, surface and volume integrals

- Line integral $\int_C F\cdot dl$ — work along C; for a **conservative** field it depends only on end points.
- Surface integral $\iint_S F\cdot dS=\iint F\cdot\hat{n}dS$ — flux through S.
- For a parametrised surface r(u,v), $dS=(\partial_u r\times\partial_v r)\,du\,dv$.

Conservative field test (on a simply connected region): $\nabla\times F=0 \iff F=\nabla\phi \iff \oint F\cdot dl=0 \iff$ line integral path-independent. **Solenoidal field:** $\nabla\cdot B=0 \iff B=\nabla\times A$.

3.5 The three integral theorems

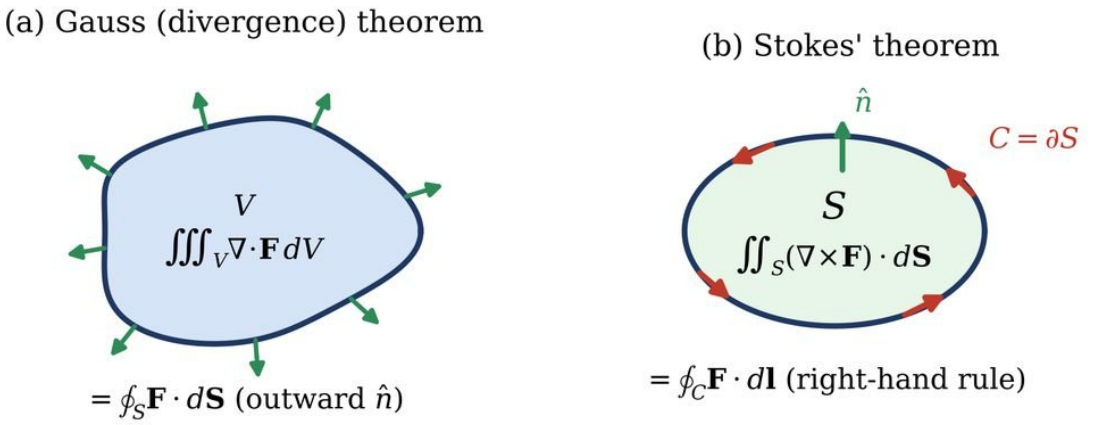


***Figure 3.3**: Gauss's theorem relates a volume integral of divergence to the outward flux; Stokes' theorem relates the flux of the curl to the circulation around the boundary, with the orientation fixed by the right-hand rule.*

Gauss: $\iiint_V \nabla\cdot F\,dV=\oint_S F\cdot dS$, Stokes: $\iint_S(\nabla\times F)\cdot dS=\oint_{\partial S} F\cdot dl$,
Green (plane): $\oint_C (P\,dx+Q\,dy)=\iint_R((\partial Q)/(\partial x)-(\partial P)/(\partial y))dx\,dy$.

Corollaries: $\oint_S \phi\, dS = \iiint_V \nabla\phi\, dV$; Green's identities $\iiint (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)dV = \oint \phi\nabla\psi\cdot dS$ and $\iiint (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \oint (\phi\nabla\psi - \psi\nabla\phi)\cdot dS$.

Worked example 3.1 (Gauss). $\mathbf{F}=\mathbf{r}$ over a sphere of radius R. Flux $= \square R\, dS = 4\pi R^3$. Volume side: $\nabla\cdot\mathbf{r}=3$ and $V=4/3\pi R^3$, so $3V=4\pi R^3$ ✓.

Worked example 3.2 (Stokes). $\mathbf{F}=(-y,x,0)$ on the unit circle in the xy-plane. $\oint \mathbf{F}\cdot d\mathbf{l} = \int_0^{(2\pi)} (\sin^2 t+\cos^2 t)dt = 2\pi$. Surface side: $\nabla\times\mathbf{F}=2\hat{\mathbf{z}}$, area $\pi \Rightarrow 2\pi$ ✓.

3.6 Dirac delta in vector calculus

Although $\nabla\cdot(\hat{\mathbf{r}}/r^2)=0$ for $r\neq 0$, the flux of $\hat{\mathbf{r}}/r^2$ through any sphere about the origin is 4π. The consistent statement is

$$\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\,\delta^3(\mathbf{r}), \qquad \nabla^2\frac{1}{r} = -4\pi\,\delta^3(\mathbf{r}).$$

This is the mathematical root of Gauss's law for a point charge and of $\Phi = q/(4\pi\varepsilon_0 r)$ solving Poisson's equation $\nabla^2\Phi = -\rho/\varepsilon_0$ (Unit 7).

3.7 Helmholtz theorem

A vector field vanishing sufficiently fast at infinity is determined uniquely by its divergence and curl: $\mathbf{F} = -\nabla\phi + \nabla\times\mathbf{A}$ with $\nabla^2\phi = -\nabla\cdot\mathbf{F}$, so specifying $\nabla\cdot\mathbf{E}$ and $\nabla\times\mathbf{E}$ (or B) fixes the field. Maxwell's four equations are precisely such specifications.

3.8 Derivations behind the operators

Gradient is normal to level surfaces. On a surface $f=c$, any displacement $d\mathbf{l}$ within it has $df = \nabla f\cdot d\mathbf{l} = 0$, so ∇f is perpendicular to the surface; the rate of change along $\hat{\mathbf{u}}$ is $|\nabla f|\cos\alpha$, maximal for $\alpha = 0$.

Divergence as outward flux per volume. For a small cube of edges dx, dy, dz centred at (x, y, z), the net outward flux through the two faces perpendicular to x is $[F_x(x+dx/2) - F_x(x-dx/2)]\,dy\,dz \approx (\partial F_x/\partial x)\,dx\,dy\,dz$. Summing three pairs of faces gives $\oint \mathbf{F}\cdot d\mathbf{S} = (\nabla\cdot\mathbf{F})\,dV$, i.e. $\nabla\cdot\mathbf{F} = \lim_{V\to 0} \frac{1}{V}\oint \mathbf{F}\cdot d\mathbf{S}$.

Gauss's theorem follows by subdividing a finite volume into such cubes: interior face contributions cancel in pairs, leaving the outer surface.

Curl as circulation per area. For a small square loop in the xy-plane, the circulation is $[F_x(y-dy/2) - F_x(y+dy/2)]dx + [F_y(x+dx/2) - F_y(x-dx/2)]dy = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)dx\,dy$, which is $(\nabla\times\mathbf{F})_z\,dA$. **Stokes' theorem** follows by tiling the surface with such loops (interior edges cancel), and **Green's theorem** is the planar case.

Why $\nabla\times\nabla f = 0$. $(\nabla\times\nabla f)_i = \varepsilon_{ijk}\partial_j\partial_k f = 0$, because ε_{ijk} is antisymmetric in jk while mixed partials are symmetric. Likewise $\nabla\cdot(\nabla\times\mathbf{F}) = \varepsilon_{ijk}\partial_i\partial_j F_k = 0$.

Area by Green's theorem. $A = \frac{1}{2}\oint (x\,dy - y\,dx)$. For the ellipse $x = a\cos t$, $y = b\sin t$: $A = \frac{1}{2}\int_0^{(2\pi)} ab\,dt = \pi ab$.

Mean value property of harmonic functions. If $\nabla^2\phi = 0$, then the value at a point equals the average over any sphere centred on it. Consequently a harmonic function has no local maximum or minimum inside a region (Earnshaw's theorem: no stable electrostatic equilibrium in free space).

3.9 Worked line, surface and flux integrals

Worked example 3.3 (path dependence). $\mathbf{F}=(y^2, x)$ from (0,0) to (1,1). Along $y=x$: $\int y^2dx + x\,dy = \int_0^1 x^2dx + \int_0^1 x\,dx = 1/3+1/2 = 5/6$. Along $y=x^2$: $\int_0^1 t^4 dt + \int_0^1 t\cdot 2t\,dt = 1/5+2/3 = (13)/(15)$. The values differ because $(\nabla\times\mathbf{F})_z = 1-2y \neq 0$: the field is not conservative.

Worked example 3.4 (flux through a hemisphere). $\mathbf{F}=\mathbf{r}$ through the upper hemisphere of radius R with outward normal: $\mathbf{F}\cdot\hat{\mathbf{n}} = R$ and area $2\pi R^2$ give $2\pi R^3$. By Gauss with the closing disc in $z=0$ (where $\mathbf{F}\cdot\hat{\mathbf{n}} = -z = 0$): $\iiint 3\,dV = 3\cdot 2/3\pi R^3 = 2\pi R^3$ ✓.

Worked example 3.5 (Stokes with a non-trivial field). $\mathbf{F}=(y,z,x)$ on the unit circle in the xy-plane. Circulation: $\square y\,dx = \int_0^{(2\pi)} \sin t(-\sin t)dt = -\pi$. Curl: $\nabla\times\mathbf{F} = (-1,-1,-1)$, so the flux through the unit disc with $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ is $-\pi$ ✓.

3.10 Scalar and vector potentials

Because $\nabla\cdot\mathbf{B}=0$ we may write $\mathbf{B} = \nabla\times\mathbf{A}$; A is defined only up to $\mathbf{A} \to \mathbf{A} + \nabla\chi$ (**gauge freedom**), which leaves B unchanged since $\nabla\times\nabla\chi = 0$. In electrostatics $\mathbf{E} = -\nabla\phi$. In electrodynamics $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$, and the Lorenz gauge $\nabla\cdot\mathbf{A} + 1/(c^2)\partial_t\phi = 0$ decouples the potentials into wave equations (Unit 8). The Coulomb gauge $\nabla\cdot\mathbf{A}=0$ is used in magnetostatics, where $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$.

3.11 Chapter problems with solutions

1. $\nabla\times(\omega\times\mathbf{r})$ **for constant** ω**.** $= \omega(\nabla\cdot\mathbf{r}) - (\omega\cdot\nabla)\mathbf{r} = 3\omega - \omega = 2\omega$, and $\nabla\cdot(\omega\times\mathbf{r}) = 0$. (The velocity field of a rigid rotation has vorticity twice the angular velocity.)
2. $\nabla(\mathbf{a}\cdot\mathbf{r})$ **for constant** $\mathbf{a}$**.** $\mathbf{a}$.
3. $\nabla^2(x^2+y^2+z^2)$**.** 6 (consistent with $\nabla^2 r^n = n(n+1)r^{(n-2)}$ at $n=2$).
4. **Unit normal to** $x^2+y^2+z=9$ **at** $(2,-1,4)$**.** $\nabla = (4,-2,1)$, $|\nabla| = \sqrt{(21)}$, $\hat{\mathbf{n}} = (0.873, -0.436, 0.218)$.
5. **Angle between** $x^2+y^2+z^2=9$ **and** $z = x^2+y^2-3$ **at** $(2,-1,2)$**.** Normals $(4,-2,4)$ and $(4,-2,-1)$: $\cos\theta = (16+4-4)/(6\cdot\sqrt{(21)}) = 0.582$, $\theta = 54.4^\circ$.
6. **Directional derivative of** $f = xy^2 + yz^3$ **at** $(1,-1,1)$ **along** $(1,2,-2)$**.** $\nabla f = (y^2,\ 2xy+z^3,\ 3yz^2) = (1,-1,-3)$; $\hat{\mathbf{u}} = (1,2,-2)/3$; $D_{\mathbf{u}}f = (1-2+6)/3 = 5/3$.
7. **Divergence and curl of** $\mathbf{F} = (x^2y,\ y^2z,\ z^2x)$**.** $\nabla\cdot\mathbf{F} = 2xy + 2yz + 2zx$; $\nabla\times\mathbf{F} = (-y^2, -z^2, -x^2)$.
8. **Outward flux of** $\mathbf{r}$ **through the unit cube** $[0,1]^3$**.** $\square 3\,dV = 3$ (each of the three faces $x=1, y=1, z=1$ contributes 1; the other three contribute 0).
9. **Show** $\ln\rho$ **is harmonic in two dimensions.** $\nabla^2\ln\rho = (1/\rho)\partial_\rho(\rho\cdot 1/\rho) = 0$ for $\rho\neq 0$ (the potential of an infinite line charge).
10. **Is** $\mathbf{F} = (2xy+z^3,\ x^2,\ 3xz^2)$ **conservative? Find its potential.** $\nabla\times\mathbf{F} = (0-0,\ 3z^2-3z^2,\ 2x-2x) = 0$, simply connected domain $\Rightarrow \mathbf{F} = \nabla\phi$ with $\phi = x^2y + xz^3$.

Traps. (1) A field with $\nabla\times\mathbf{F}=0$ is conservative only on a simply connected region: $\mathbf{F} = (-y,x,0)/(x^2+y^2)$ has zero curl away from the z-axis, but $\oint \mathbf{F}\cdot d\mathbf{l} = 2\pi$ around it. (2) The sign of $d\mathbf{S}$ on a closed surface is outward. (3) ∇^2 of a vector in curvilinear coordinates is not the scalar Laplacian applied componentwise; use $\nabla(\nabla\cdot\mathbf{F}) - \nabla\times\nabla\times\mathbf{F}$. (4) $\nabla\cdot(\hat{\mathbf{r}}/r^2)$ is zero everywhere except the origin, where it is $4\pi\delta^3$.

One-look revision (Chapter 3). $\nabla\times\nabla f = 0$; $\nabla\cdot\nabla\times\mathbf{F} = 0$; $\nabla\cdot\mathbf{r} = 3$; $\nabla^2(1/r) = -4\pi\delta^3$; Gauss, Stokes, Green; conservative $\Leftrightarrow$ curl-free on simply connected domain; spherical ∇^2 radial part $r^{(-2)}\partial_r(r^2\partial_r)$.