Roll No. ........................

(05/24)

# 11658

## M. Sc. (2 Year) EXAMINATION

(For Batch 2019 to 2020 Only)

(Second Semester)

MATHEMATICS

MTHCC-2201

Advanced Abstract Algebra

*Time : Three Hours* *Maximum Marks : 70*

**Note :** Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory. All questions carry equal marks. Marks are indicated along with questions.

### Compulsory Question

1. (a) Define a prime subfield. **2**

(b) Show that $x^2+3$ and $x^2+x+1$ have same splitting field. **2**

(c) Prove that no irreducible polynomial over a field of characteristic zero has multiple root in any field extension. 2

(d) Let char. (F) = $p$. Suppose $f(x) = x^p - x$. Show that all the roots of $f(x)$ in its splitting field over F are distinct. 2

(e) If [K : F] = 2, prove that K is normal extension of F. 2

(f) Prove that any constructible complex number is algebraic over Q of degree a power of 2. 2

(g) Under usual notations define the companion matrix of $f(x)$ i.e. $c(f(x))$, where : 2

$$f(x) = x^k - \alpha_{k-1}x^{k-1} - \dots\dots - \alpha_1 x - \alpha_0$$

## Unit I

2. (a) Prove that any prime field is either isomorphic to the field of rational numbers or to the field of integers modulo some prime number. 7



(b) If L is an algebraic extension of K and K is an algebraic extension of F, then L is an algebraic extension of F. 7

3. (a) If $a$, $b$ $\in$ K are algebraic over F of degrees $m$ and $n$ respectively, where $m$ and $n$ are relatively prime; prove that F($a$, $b$) is of degree $mn$ over F. 7

(b) Find the splitting field and degree of the splitting of the polynomial : 7

$$x^3 - 3x + 1$$

## Unit II

4. (a) Prove that any algebraic extension of a finite field F is a separable extension. 7

(b) Let D be an integral domain of char.(F) = $p$, a prime number. Then prove that :

(i) The mapping $\sigma : D \to D$ s.t. $\sigma(a)$ $a^p$ for $a \in D$ is a monomorphism.

(ii) For any positive integer $n$, the mapping $\sigma_n : D \to D$ s.t. $\sigma_n(a) = ap^n$ for $a \in D$ is a monomorphism. 7

5. (a) Prove that $\phi_n(x)$ is irreducible over Q. 7

(b) Let K be a finite algebraic extension of a field F. Then K is a normal extension of F iff K is the splitting field over F of some non-zero polynomial over F. 7

## Unit III

6. Let G be a finite group of automorphisms of a field K; $F_0$, the fixed field under G.

Then the degree of K over $F_0$ is equal to the order of the group G i.e. $[K : F_0] = o(G)$. 14

7. (a) Prove that in general an angle cannot be trisected by ruler and compass. 7

(b) Find the Galois group of the splitting field of the polynomial $x^4 - 2$ over Q.

7



## Unit IV

8. Let dim(V) = $n$ and T ∈ A(V) is nilpotent and its index of nilpotency is $n_1$, then ∃ a basis of V s.t. the matrix of T in this basis is of the form :

$$\begin{bmatrix} M_{n_1} & 0 & \ldots\ldots & 0 \\ 0 & M_{n_2} & \ldots\ldots & 0 \\ 0 & 0 & M_{n_2} & 0 \\ 0 & 0 & \ldots\ldots & M_{n_r} \end{bmatrix}_{n\times n}$$

where $n_1 \geq n_2 \geq \ldots\ldots \geq n_r$ and $n_1 + n_2 + \ldots\ldots + n_r = n$ and 14

$$M_k = \begin{bmatrix} 0 & 1 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 1 & \ldots\ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots\ldots & 1 \\ 0 & 0 & 0 & \ldots\ldots & 0 \end{bmatrix}_{k\times k}$$

9. Suppose T ∈ A(V) be nilpotent then prove that invarients of T are unique. 14