Roll No. ....

(05/24)

15212

M.Sc. (2 Year) EXAMINATION

(For Batch 2021 & Onwards)

(Second Semester)

MATHEMATICS

MSc/Math/2/CC7

Measure and Integration Theory

Time : Three Hours *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory.

Compulsory Question

1. (a) Define a measurable set and prove that ϕ and R are measurable.

(b) Prove that a function is measurable iff both its positive and negative parts are measurable functions.

(c) Enumerate four shortcomings of Riemann integral as compare to Levesgue integral.

(d) Let $fn(x) = nx^n$, $0 \leq x \leq 1$. Then

$$\lim_{h\to\infty}\int_0^1 fn \neq \int_0^1 \lim_{n\to\infty} fn.$$

(e) Prove that a bounded monotone function is a function of bounded variation and $T_a^b(f) = |f(b) - f(a)|$. 2×5=10

Unit I

2\. (a) Let $\{E_n\}$ be a countable collection of sets. Then $m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n)$. Use this result to show that a countable set has measure zero.

(b) If E_1 and E_2 one measurable sets, then $E_1 \cup E_2$ and $E_1 \cap E_2$ one measurable. 15

3\. (a) Let $\{E_i\}$ be an infinite increasing sequence of sets (not necessarily measurable). Then :

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m^*(E_n).$$

(b) Prove that there exists a non-measurable set in the interval ([0, 1)]. 15

Unit II

4\. (a) Ler f and g be measurable functions. Then show that $f + g$, $f - g$, $|f|$, f^2 are measurable.

(b) If a function f defined on E is continuous *a.e.*, then f is measurable on E. 15

5\. (a) State and prove Egoroff's theorem.

(b) Let $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$. Then :

(i) $f_{n+g_n} \xrightarrow{m} f + g$

(ii) $\alpha f_n \xrightarrow{m} \alpha f$, α is a real number

(iii) $f_n^+ \xrightarrow{m} f^+, f_n^- \xrightarrow{m} f^-$ and

$|f_n| \xrightarrow{m} |f|$. 15

Unit III

6. (a) Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then :

$\int(a\phi + b\psi) = a\int\phi + b\int\psi$, for all reals a and b.

(b) State and prove Bounded convergence theorem. Does the result true in case of Riemann integrals ? 15

7. (a) Let f and g be integrable functions over E. Then $(f + g)$ is integrable over E and

$\int_E (f+g) = \int_E f + \int_E g$.

(b) Show that the existence of an integrable 'dominant function g' in Lebesgue dominated convergence theorem is sufficient but not necessary for the interchange of the limit and integral operations. 15

Unit IV

8. Let f be an increasing real-valued function defined on $[a, b]$. Then f is differentiable *a.e.* and the derivative f' is measurable.

Further more $\int_a^b f'(x)dx \le f(b) - f(a)$. 15

9. (a) Let f be a bounded and measurable function defined on $[a, b]$. If $F(x) = \int_a^x f(t)dt + F(a)$ then $f'(x) = f(x)$ *a.e.* in $[a, b]$.

(b) Let f be a function of bounded variation on $[a, b]$. Then f is absolutely continuous on $[a, b]$ if the variation function V_f, given by $V_f(x) = T_a^x(f)$, is absolutely continuous on $[a, b]$. 15