Roll No. ....

(12/24)

15226

M. Sc. (2 Year) EXAMINATION

(For Batch 2021 & Onwards)

(Third Semester)

MATHEMATICS

MSc/Maths/3/DSC11

Number Theory

Time : Three Hours *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory.

1. (i) For any real number x prove that $[x]+\left[x+\frac{1}{2}\right]=[2x]$. 3

(ii) The equation $15x^2 - 7y^2 = 9$ has no solution in integers. 3

(iii) Let a/b, a'/b', a''/b'' be any three consecutive fractions in the Farey sequence of order n. Prove that :

$$a'/b' = (a + a'')/(b + b''). \quad 3$$

(iv) Define unimodular matrices and ternary quadratic forms. 3

(v) Expand the rational fractions 17/3 and 3/17 into finite simple continued fractions. 2

Unit I

2. (a) Let p denote a prime. Then, prove that the Largest exponent e such that $p^e/n!$ is :

$$e = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right]. \quad 7$$

(b) Find a formula for :

$$u_n = 2u_{n-1} - u_{n-2}, u_0 = 0, u_1 = 1.$$

Also, if $u_0 = 1$ and $u_1 = 1$. 7

3. (a) Prove that : 7

$$\phi(n) = \sum_{d/n} \mu(d)\left(\frac{n}{d}\right) = \sum_{d/n} d\,\mu\left(\frac{n}{d}\right).$$

(b) Prove that :

$$d(n) = \sum_{d/n} 1$$

is a multipliative function. Also find a formula for $d(n)$. 7

Unit II

4. (a) Find all integers x and y such that $147x + 258y = 369$. 7

(b) Prove that the positive primitive solutions of :

$$x^2 + y^2 = z^2$$

with y even are

$$x = r^2 - s^2, y = 2rs, z = r^2 + s^2$$

where r and s are arbitrary intergers of opposite parity with $r > s > 0$ and $(r, s) = 1$. 7

5. (a) Find all solutions of the simultaneous congruences : 7

$$3x + 3z \equiv 1 \pmod 5,\ 4x - y + z \equiv 3 \pmod 5.$$

(b) Find all rational points on the ellipse $x^2 + 5y^2 = 1$. 7

Unit III

6. (a) If a/b and a'/b' are consecutive fractions in any row, then prove that among all rational fractions with values between these two $(a+a')/(b+b')$ is the unique fraction with smallest denominator. 7

(b) If δ is a real and irrational, then prove that there are infinitely many distinct rational numbers $\frac{a}{b}$ such that

$$\left|\delta - \frac{a}{b}\right| < \frac{1}{b^2}.$$ 7

7. State and prove Lagrange's four square theorem. 14

Unit IV

8. (a) If $(a_0, a_1, ..., a_j) = (b_0, b_1, ..., b_n)$ where these finite continued fractions are simple, and if $a_j > 1$ and $b_n > 1$, then $j = n$ and $a_i = b_i$ for i = 0, 1, ..., n. 7

(b) Prove that $x^2 - dy^2 = -1$ has no solution if $d \equiv 3 \pmod 4$. 7

9. Prove that the continued fraction expansion of the real quadratic irrational number δ is purely periodic if and only if $\delta > 1$ and $-1 < \delta' < 0$, where δ' denotes the conjugate of δ. 14