Roll No. ....

(05/24)

15237

M. Sc. (2 Year) EXAMINATION

(For Batch 2021 & Onwards)

(Fourth Semester)

MATHEMATICS

MSc/Maths/4/DSC20

Algebraic Number Theory

Time : Three Hours *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory.

1\. (a) Define integral basis.

(b) Define Dedekind domains.

(c) Define L.C.M. of ideals in O_K.

(d) Define Hurwitz constant.

(e) Define Legendre symbol. 10

Unit I

2. (a) If K is an algebraic number field of degree n over Q and $\alpha \in O_K$ its ring of integers, then prove that $Tr_K(\alpha)$ and $N_K(\alpha)$ are in Z. 8

(b) Show that O_K has an integral basis. 7

3. (a) Suppose K is a number field with r_1 real embeddings and $2r_2$ complex embeddings so that $r_1 + 2r_2 = [K:Q] = n$. Show that d_K has sign $(-1)^{r_2}$. 8

(b) Let $K = Q(\alpha)$, where $\alpha = r^{1/3}$, $r = ab^2 \in Z$, ab is squarefree. If $3/r$, assume that $3/a$, $3/b$, find an integral basis for K. 7

Unit II

4. (a) Show that every non-zero prime ideal P of O_K is maximal. 8

(b) Show that any principal ideal domain is a Dedekind domain. 7

5. (a) Let P be a prime ideal of O_K, D the different of K. If $P^e \mid \langle p \rangle$, where $p \in Z$ is a prime. Then, prove that $P^{e-1} \mid D$.

(b) State and prove Dedekind theorem. 7

Unit III

6. (a) Show that given $\alpha, \beta \in O_K$, there exists $t \in Z$, $|t| \leq H_K$ and $w \in O_K$ so that :

$$|N(\alpha t - \beta w)| < |N(\beta)|.$$ 8

(b) Prove that the number of equivalence classes of ideals is finite. 7

7. (a) Compute the class number of $Q(\sqrt{-7})$. 8

(b) Let K be an algebraic number field of degree n over Q. Show that each ideal class contain an ideal A satisfying :

$$N_A \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2} |d_K|^{1/2},$$

where r_2 is the number of pairs of complex embeddings of K and d_K is the descriminant. 7

Unit IV

8. (a) For all odd primes p, show that :

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8 \\ -1 & \text{if } p \equiv 3, 5 \pmod 8 \end{cases}$$ 8

(b) Show that $S^q \equiv \left(\frac{q}{p}\right) S \pmod q$, where q and p are odd primes, S is Gauss sum for prime p. 7

9. (a) State and prove quadratic law of reciprocity for Legendre symbols. 8

(b) Show that there are infinite many primes of the type $4k+1$. 7