Roll No. ....

(12/24)

15231

M.Sc. (2 Year) EXAMINATION

(For Batch 2021 & Onwards)

(Fourth Semester)

MATHEMATICS

MSc/Maths/4/CC12

Functional Analysis

Time : Three Hours *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory.

1. (a) What do you mean by B(N, N′) ? Also explain B(N, C).

   (b) Prove that every infinite dimensional subspace need not be closed.

(c) Give an example of a space which is not reflexive.

(d) If H is Hilbert space then prove that :

$$\| x+y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + \| y \|^2 \quad \forall x, y \in H.$$

(e) If A_1 and A_2 are self adjoint operators on H, then prove that $A_1 A_2$ is self adjoint if and only if $A_1 A_2 = A_2 A_1$. 10

Unit I

2. (a) Let X be a linear space over a field F and d be the metric on X such that $d(x, y) = d(x - y, 0)$ and $d(\alpha x, 0) = |\alpha| d(x, 0) \forall x, y \in X$ and $\alpha \in F$. Define $\|x\| = d(x, 0) \ \forall x \in X$. Prove that $\|\cdot\|$ is a norm of X and d is the metric induced by $\|\cdot\|$ on X. 7.5

(b) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset x + M in the quotient space N/M is defined by $\|x + M\| = \inf\{\|x + m\|; m \in M\}$. Then show that N/M is a Banach space if N is a Banach space. 7.5

3. (a) Let C(X) denote the linear space of all bounded continuous scalar valued functions defined on a topological space X. Show that C(X) is Banach space under the norm $\| f \| = \sup \{| f(x)|, x \in X\}; f(x) \in C(X)$ is complete. 7.5

(b) Let N and N′ be normed linear spaces and let T be a linear transformation of N into N′. Then the inverse T^{-1} exists and is continuous on its domain of definition if and only iff there exists a constant $m > 0$ such that $m \| x \| \leq \| T(x) \|; \forall x \in N$. 7.5

Unit II

4. (a) If N is a normed linear space and x_0 is a non-zero vector in N, then prove that there exists a functional f_0 in N* such that $f_0(x_0) = \| x_0 \|$ and $\| f_0 \| = 1$. In particular, if $x \neq y (x, y \in N)$, there exists a vector $f \in N^*$ such that $f(x) \neq f(y)$. 7.5

(b) Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M, then prove that there exists a functional F in N* such that F(M) = $\{x_0\}$ and $F(x_0) \neq 0$. 7.5

5. (a) State and prove Uniform boundedness theorem. 7.5

(b) Prove that a non-empty subset X of a normed linear space N is bounded iff f (X) is a bounded set of numbers for f in N*. 7.5

Unit III

6. (a) Prove that in a finite dimensional space, the notion of weak and strong convergence are equivalent. 7.5

(b) Let B and B′ are Banach spaces and let $T : B \to B'$ be a linear transformation then prove that graph of T is closed iff T is continuous. 7.5

7. (a) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non-zero vector z_0 in H such that $z_0 \perp M$. 7.5

(b) Give an example of a Banach space which is not Hilbert space. 7.5

Unit IV

8. (a) If $< e_i >$ is an orthonormal set in Hilbert space H, then prove that $\sum |(x, e_i)|^2 \leq \|x\|^2$ for every vector $x \in H$. 7.5

(b) Prove that every non-zero Hilbert space contains a complete orthonormal set. 7.5

9. (a) If N_1 and N_2 are normal operators on a Hilbert space H with property that either commutes with the adjoint of the other, then prove that $N_1 + N_2$ and N_1N_2 are normal. 7.5

(b) Prove that an operator T on H is unitary if and only if it is an isometric isomorphism of H into itself. 7.5