Roll No. ....

(05/24)

11671

M.Sc. (2 Year) EXAMINATION

(For Batch 2017 to 2020 Only)

(Fourth Semester)

MATHEMATICS

MTHCC-2401

Functional Analysis

Time : Three Hours *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory. All questions carry equal marks.

Compulsory Question

1. (a) Show that $||.||$ is continuous as a mapping from a normed space X into R.

(b) Let X and Y be normed spaces. Define the function $A : X \rightarrow Y$ by $Ax = 0 \; \forall x \in X$, where 0 is the zero vector in Y. Show that A is a linear operator and $\|A\| = 0$.

(c) Define adjoint of a bounded linear operator and show that adjoint is also a linear operator.

(d) Define a reflexive space. Whether R^n is reflexive or not.

(e) Define weakly convergent sequence in a normed space and show that weak limit is unique.

(f) Show that in an inner product space, $< x, \alpha y + \beta z > = \bar{\alpha} < x, y > + \bar{\beta} < x, z >$.

(g) Define orthonormal set and write the orthogonal set of R^3.

Unit I

2. (a) Prove that the Euclidean space R^n is Banach space.

(b) A normed space X is finite dimensional iff the closed unit sphere in X is compact.

3. (a) Let X and Y be normed space over the field k and $T : X \rightarrow Y$ a linear operator. Then T is continuous iff T is bounded.

(b) Show that dual space of $l^p(n)$, $1 < p < \infty$ is $l^q(n)$, where $1 < q < \infty$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Unit II

4. (a) Let X be a normed space over the field k, M a subspace of X and let $x_0 \in X$ be such that $d(x_0, M) = d > 0$. Then $\ni$ a $g \in X^*$ s.t.

(i) $g(x_0) = 1$. (ii) $g(M) = 0$

(b) State and prove Riesz representation theorem for bounded linear functionals on $C[a, b]$.

5. (a) State and prove uniform boundedness theorem.

(b) Let X and Y be normed spaces once the field k and $T : X \to Y$ a bounded linear operator. Then :

(i) $\|T^*\| = \|T\|$

(ii) The mapping $T \to T^*$ is an isometric isomorphism of B(X, Y) into B(Y*, X*).

Unit III

6. (a) Let $\{x_n\}$ be a sequence in a normal space X. Then $x_n \to x$ in $X \Rightarrow x_n \xrightarrow{w} x$ in X. Discuss the converse also.

(b) State and prove bounded inverse theorem.

7. (a) State and prove Schwartz inequality and use it prove triangle inequality in an inner product space.

(b) Let X be an inner product space and $M \neq \phi$ be a complete proper subspace of X. Then $M^{\perp} \neq \phi$.

Unit IV

8. (a) State and prove Bessel's inequality in an inner product space.

(b) State and prove Parseval's identity.

9. (a) State and prove Riesz representation theorem for bounded linear functionals on a Hilbert space.

(b) Let N(H) be the set of all normal operators on H. Further let S, T $\in$ N(H) and suppose that ST* = T* S. Then S + T and ST are in N(H).