Roll No. ....

(05/25)

15237

M. Sc. (2 Year) EXAMINATION

(For Batch 2021 & Onwards)

(Fourth Semester)

MATHEMATICS

MSc/Maths/4/DSC20

Algebraic Number Theory

Time : Three Hours *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory.

1. (a) Define norm and traces. 2

(b) State Hurwitz lemma. 2

(c) Compute the class number of $Q(\sqrt{21})$. 2

(d) Show that the sum and product of two fractional ideals are again fractional ideals. 2

(e) State law of quadratic reciprocity. 2

Unit I

2. (a) Suppose that the minimal polynomial of α is Eisensteinian with respect to a prime p. Show that the index of α is not divisible by p. 8

(b) Show that $\det\left(Tr\left(w_i w_j\right)\right)$ is independent of the choice of integral basis. 7

3. (a) If $D \equiv 1 \pmod 4$, show that every integer of $Q(\sqrt{D})$ can be written as $\left(a+b\sqrt{D}\right)/2$, where $a \equiv b \pmod 2$. 8

(b) Show that every nonzero prime ideal in O_k contains exactly one integer prime. 7

Unit II

4. (a) Let $\wp$ be a prime ideal of O_k. Then, prove that there exists $\in K$, $z \notin O_k$, such that $z\wp \subseteq O_k$. 8

(b) State and prove Chinese Remainder Theorem. 7

5. (a) Let L/K be a finite extension of algebraic number fields. Show that the map $Tr_{L/K}: L \times L \to K$ is non-degenerated. 8

(b) Let D be the different of an algebraic number field K. Then, prove that $N(D) = |d_K|$. 7

Unit III

6. (a) Show that the class number of $K = Q(\sqrt{D})$, for $D = -1, -2, -3, -7$ are 2. 8

(b) Show that the equation $x^2 + 5 = y^3$ has no integral solution. 7

7. (a) Show that there exist bounded, symmetric convex domains with volume $< 2^n$ that do not contain a lattice point. 8

(b) State and prove Minkowski Bound. 7

Unit IV

8. (a) For all odd primes p, prove that :

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8 \\ -1 & \text{if } p \equiv 3, 5 \pmod 8 \end{cases}$$ 8

(b). Show there are an infinite number of primes in the arithmetic progression $15k + 4$. 7

9. (a) For and odd prime p, prove that the Gauss sum S satisfies : 8

$$S^2 = \left(\frac{-1}{p}\right) p.$$

(b) Show that $x^4 \equiv 25 \pmod{1013}$ has no solution. 7

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