

Methods of Real Analysis

Richard R Goldberg



Contents

<i>Introduction: assumptions and notations</i>	1
1. Sets and functions	3
1.1. Sets and elements	3
1.2. Operations on sets	4
1.3. Functions	7
1.4. Real-valued functions	11
1.5. Equivalence. Countability	14
1.6. Real numbers	18
1.7. Least upper bounds	21
2. Sequences of real numbers	24
2.1. Definition of sequence and subsequence	24
2.2. Limit of a sequence	26
2.3. Convergent sequences	30
2.4. Divergent sequences	32
2.5. Bounded sequences	34
2.6. Monotone sequences	35
2.7. Operations on convergent sequences	38
2.8. Operations on divergent sequences	44
2.9. Limit superior and limit inferior	45
2.10. Cauchy sequences	52
2.11. Summability of sequences	52
2.12. Limit superior and limit inferior for sequences of s	55

3.	<i>Series of real numbers</i>	
3.1.	Convergence and divergence	67
3.2.	Series with nonnegative terms	67
3.3.	Alternating series	69
3.4.	Conditional convergence and absolute convergence	71
3.5.	Rearrangements of series	73
3.6.	Tests for absolute convergence	76
3.7.	Series whose terms form a nonincreasing sequence	80
3.8.	Summation by parts	85
3.9.	(C, 1) summability of series	88
3.10.	The class l^2	90
3.11.	Real numbers and decimal expansions	93
		96
4.	<i>Limits and metric spaces</i>	
4.1.	Limit of a function on the real line	98
4.2.	Metric spaces	98
4.3.	Limits in metric spaces	105
		106
5.	<i>Continuous functions on metric spaces</i>	
5.1.	Functions continuous at a point on the real line	113
5.2.	Reformulation	113
5.3.	Functions continuous on a metric space	116
5.4.	Open sets	118
5.5.	Closed sets	121
5.6.	Discontinuous functions on R^1	124
		128
6.	<i>Connectedness, completeness, and compactness</i>	
6.1.	More about open sets	133
6.2.	Connected sets	133 o
6.3.	Bounded sets and totally bounded sets	134
6.4.	Complete metric spaces	138
6.5.	Compact metric spaces	141
6.6.	Continuous functions on compact metric spaces	145
6.7.	Continuity of the inverse function	148
6.8.	Uniform continuity	150
		152

Contents

7. Calculus	156
7.1. Sets of measure zero	156
7.2. Definition of the Riemann integral	157
7.3. Existence of the Riemann integral	163
7.4. Properties of the Riemann integral	165
7.5. Derivatives	170
7.6. Rolle's theorem	177
7.7. The law of the mean	181
7.8. Fundamental theorems of calculus	183
7.9. Improper integrals	189
7.10. Improper integrals (continued)	196
8. The elementary functions. Taylor series	202
8.1. Hyperbolic functions	202
8.2. The exponential function	204
8.3. The logarithmic function. Definition of x^a	206
8.4. The trigonometric functions	203
8.5. Taylor's theorem	214
8.6. The binomial theorem	221
8.7. L'Hospital rule	222
9. Sequences and series of functions	231
9.1. Pointwise convergence of sequences of functions	231
9.2. Uniform convergence of sequences of functions	234
9.3. Consequences of uniform convergence	238
9.4. Convergence and uniform convergence of series of functions	243
9.5. Integration and differentiation of series of functions	247
9.6. Abel summability	250
9.7. A continuous, nowhere-differentiable function	256
10. Three famous theorems	259
10.1. The metric space $C[a, b]$	259
10.2. The Weierstrass approximation theorem	261
10.3. Picard existence theorem for differential equations	260
10.4. The Arzela theorem on equicontinuous families	268

11. <i>The Lebesgue integral</i>	
11.1. Length of open sets and closed sets	271
11.2. Inner and outer measure. Measurable sets	271
11.3. Properties of measurable sets	274
11.4. Measurable functions	278
11.5. Definition and existence of the Lebesgue integral for bounded functions	283
11.6. Properties of the Lebesgue integral for bounded measurable functions	285
11.7. The Lebesgue integral for unbounded functions	294
11.8. Some fundamental theorems	300
11.9. The metrix space $L^2[a, b]$.	308
11.10. The integrals on $(-\infty, \infty)$ and in the plane	315
12. <i>Fourier series</i>	328
12.1. Definition of Fourier series	328
12.2. Formulation of convergence problems	331
12.3. The $(C, 1)$ summability of Fourier series	335
12.4. The L^2 theory of Fourier series	337
12.5. Convergence of Fourier series	342
12.6. Orthonormal expansions in $L^2[a, b]$	347
INDEX OF SPECIAL SYMBOLS	355
INDEX	357